Information Reduction in Credit Risk Models

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Abstract

This paper builds the mathematical foundation for credit risk models with incomplete information. We provide rigorous mathematical definitions for the continuously and discretely delayed filtrations. Our definitions unify two well-known types of incomplete information: the noisy information in Duffie and Lando (2001) and partial information in Collin-Dufresne et al. (2004).

Under this framework, we first show how delayed information sets, such as those available to market participants, transform structural models into reduced-form intensity-based models, via obtaining explicit analytic expressions for default intensities under various models. We then establish relations of risk debt pricings under complete and delayed filtrations.

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1 Introduction

Given the size and rapid growth of the credit risk market in recent years (see Creditflux, 2004), it is not surprising that the credit risk literature has experienced similar growth, as evidenced by the number of books and articles being published on this topic. Two types of models are studied in the credit risk literature: structural and reduced-form. Structural models view a firm’s liabilities as complex put options on the firm’s assets. Therefore, modelled in this approach are the firm’s liability structure and the firm’s asset value process. This methodology originated with Black and Scholes (1973) and Merton (1974). In these models, the default time is usually characterized as the first hitting time of a firm’s asset value to a given boundary, the boundary being determined by the firm’s liabilities. As such, if the firm’s asset value process follows a diffusion, then the default time is usually a predictable stopping time. The difficulty of using the structural approach is twofold: first, the firm’s asset value process is not directly observable, making empirical implementation difficult; and second, a predictable default time implies credit spreads should be near zero on short maturity corporate debt. This second implication is well known to be inconsistent with historical market credit spread data.

In contrast, the reduced-form approach was developed precisely to avoid directly modelling the firm’s unobservable asset value process. This was instead accomplished by modelling the price process of the firm’s liabilities, for example, a zero-coupon bond issued by the firm. This approach was originated by Jarrow and Turnbull (1992, 1995), Artzner and Delbaen (1995), and Duffie and Singleton (1999). Typically, reduced-form models characterize default as the first jump time of a point process, often assumed to follow a Cox process (i.e., a doubly stochastic Poisson process). As such, the default time is usually a totally inaccessible stopping time, implying non-zero credit spreads for short maturity corporate debt. A review of the credit risk literature can be found in many good books, including Ammann (2001), Bielecki and Rutkowski (2002), Duffie and Singleton (2003), and Lando (2004). A systematic study of the mathematical tools in reduced-form models is available in Elliott, Jeanblanc, and Yor (2000) and Jeanblanc and Rutkowski (2002).

To summarize, structural and reduced-form models are viewed as competing paradigms. However, recent work by Duffie and Lando (2001), Collin-Dufresne, Goldstein, and Helwege (2003), Çetin et al. (2004) and Jarrow and Protter (2004) point out an intrinsic connection between these two approaches. Reduced-form models can be viewed as structural models that are analyzed under different filtrations. Structural models are based on the information set available to the firm’s management, which includes continuous-time observations of both asset values and liabilities. Reduced-form models are based on the information set available to the market, typically including only partial observations of both the firm’s asset values and liabilities. As shown in examples by Duffie and Lando (2001), Collin-Dufresne, Goldstein, and Helwege (2003), Çetin et al. (2004), and Jarrow and Protter (2004), it is possible to transform a structural model with a predictable default time into a reduced-form model, with a totally inaccessible default time, by formulating the so called “incomplete information”. For instance, Duffie and Lando (2001) used noisy and discretely observed asset value in a continuous-time model, Collin-Dufresne, Goldstein, and Helwege (2003) used a simple form of delayed information in a Brownian motion type model.

However, up to date, the notion of “incomplete information” has not been mathematically and systematically studied. Indeed, it has not been even well-defined mathematically, despite the fast growing literature of information-based credit risk studies. Furthermore, it is unclear if incomplete information such as the “noisy information” and the “delayed information” can be unified under a proper mathematical framework.
Our paper is to address this issue. We define rigorously the notion of “delayed information”, for both discrete and continuous type. The latter is built on the work of Jacod and Skorohod (1994) on jumping filtrations of one sequence of marked point processes, while the former is defined through a time change of filtrations. We prove the necessity of differentiating these two types by showing that they are both intuitively and technically distinct. We illustrate via examples the generality of our definitions.

Built on this mathematical framework, we generalize the work by Duffie and Lando (2001), Collin-Dufresne et al. (2003), and characterize the existence of an intensity process for any Markov models, with and without jumps. As in earlier work, we show that delayed information transforms a predictable default time into a totally inaccessible stopping time. Moreover, we derive explicit analytical connections between the default intensity and the density function of the corresponding first passage time for general continuous Markov models. In addition, we provide a characterization of the intensity process that is useful for empirical estimation (see Chava and Jarrow (2004) and Duffie and Wang (2003) for existing empirical studies).

An outline of this paper is as follows. Section 2 builds the mathematical framework of incomplete information. First we define continuously delayed and discretely delayed filtrations, then provide general results characterizing the difference of these two types of incomplete information. We then illustrate via examples how “noisy” and “partial” information are unified under this definition. Section 3 characterizes the relation of delayed information with intensity process: this is shown by studying representative models for the asset value process: one-dimensional continuous (strong) Markov models, regime switching, and jump diffusion. Explicit formulas for the default intensities are derived via Meyer’s Laplacian approximation, as in Duffie and Lando (2001). Section 4 studies the relations of risky debt pricings under different filtrations: complete and incomplete.

2 Delayed filtration: Mathematical definition

Intuitively, the continuously delayed filtration allows information to flow continuously, but following a time clock slower than the ordinary one. The discretely delayed filtration, on the other hand, does not allow new information to flow in between two consecutive observation times. Therefore, these two types of filtration delay are distinct.

Delayed filtration (continuously delayed).

\textbf{Definition 1.} Given a filtration $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ that satisfies the usual hypotheses, i.e. it is right continuous and contains all the negligible sets. Suppose that $(\alpha_t)_{t \geq 0}$ is a time change of $\mathbb{H}$ \footnote{That is, $(\alpha_t)_{t \geq 0}$ is an increasing, right continuous process, so that $\alpha_0 = 0$ and $\alpha_t$ is an $\mathbb{H}$-stopping time for every $t \geq 0$.} so that for a.s. $\omega$, $\alpha_t(\omega) \leq t$ for all $t$. The time-changed filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0} = (\mathcal{H}_{\alpha_t})_{t \geq 0}$ is called a continuously delayed filtration of $\mathbb{H}$.

It’s not hard to show that $\mathbb{F} = (\mathcal{H}_{\alpha_t})_{t \geq 0}$ is well-defined and satisfies the usual hypotheses. See, for example, He, Wang and Yan (1992, Chapter III, §5).

\textbf{Example 1.} Given a filtration $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ and a constant $\delta > 0$. Let $\alpha_t = (t-\delta)^+$, then $\mathcal{F}_t = \mathcal{H}_{\alpha_t}$.
defines a continuously delayed filtration of $\mathbb{H}$, so that

$$\mathcal{F}_t = \begin{cases} 
\mathcal{H}_0, & t \leq \delta, \\
\mathcal{H}_{t-\delta}, & t > \delta.
\end{cases}$$

This includes as a special case the example in Collin-Dufresne, Goldstein, and Helwege (2003, Appendix A1), where the filtration $\mathbb{H}$ is the natural filtration of Brownian motion.

Delayed filtration (discretely delayed).

**Definition 2.** Given a filtration $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ that satisfies the usual hypotheses. Suppose $(T^k_n)_{n \geq 1}$ $(1 \leq k \leq K)$ are $K$ strictly increasing\(^2\) sequences of $\mathbb{H}$-stopping times. Suppose also $(\mathcal{G}_{i_1 \ldots i_K})_{i_1, \ldots, i_K \in \mathbb{N}}$ is a family of sub $\sigma$-field of $\mathcal{H}_\infty = \bigvee_{t \geq 0} \mathcal{H}_t$, so that

(i) $\mathcal{G}_{i_1 \ldots i_K} \subset \mathcal{G}_{j_1 \ldots j_K}$, if $i_1 \leq j_1, \ldots, i_K \leq j_K$;

(ii) $\mathcal{G}_{i_1 \ldots i_K} \cap \mathcal{H}_{\max_{i_1} \ldots \max_{i_K}} \cap \mathcal{H}_{\max_{i_1} \ldots \max_{i_K}}$, where $T^1_{i_1} \lor \cdots \lor T^K_{i_K} = \max\{T^1_{i_1}, \ldots, T^K_{i_K}\}$;

(iii) For any $k$, $T^k_n$ is $\mathcal{G}_{j_1 \ldots j_K}$-measurable, if $n \leq j_k$.

Define

$$\mathcal{F}^0_t = \bigcup_{i_1 \ldots i_K} (\mathcal{G}_{i_1 \ldots i_K} \cap \{T^k_{i_k} \leq t < T^k_{i_{k+1}} : 1 \leq k \leq K\}),$$

and

$$\mathcal{F}_t = \bigcup_{i_1 \ldots i_K} ((\mathcal{G}_{i_1 \ldots i_K} \lor \sigma(N)) \cap \{T^k_{i_k} \leq t < T^k_{i_{k+1}} : 1 \leq k \leq K\}),$$

where $N$ is the collection of all negligible sets. Then $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is a filtration that satisfies the usual hypotheses and is called a discretely delayed filtration of $\mathbb{H}$.

**Remark 1.** Discretely delayed filtration arises naturally from discrete observations. For instance, in Duffie and Lando (2001), a firm’s asset value is observed (with noise) at a deterministic and discrete set of times. A natural generalization of their model is that the sequence of observation times can be random as well: the firm’s asset value is updated when it issues a press release, or articles in the financial press. Moreover, there could be more than one sequence of observation times.

Mathematically speaking, the above definition is the generalization of filtration generated by a marked point process defined in Jacod and Skorohod (1994) or filtration of the discrete type as in He, Wang, and Yan (1992). It is essentially the filtration generated by finitely many marked point processes.

To see Definition 2 is well defined, we have the following lemma.

**Lemma 1.** Using the notation in Definition 2, we have

1) $\mathbb{F}^0 = (\mathcal{F}^0_t)_{t \geq 0}$ is a right-continuous sub-filtration of $\mathbb{H}$;

2) Each $T^k_n$ is a stopping time of $\mathbb{F}^0$;

3) $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is the usual augmentation of $\mathbb{F}^0$.

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\(^2\)A sequence $(T^k_n)_{n \geq 0}$ of random variables is strictly increasing, if $T^k_0 = 0$, $T^k_n \uparrow \infty$ and $\forall n \geq 0$, $T^k_n < \infty \implies T^k_n < T^k_{n+1}$. 

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Proof. 1) First, we check $\mathcal{F}_t^0$ is indeed a $\sigma$-field.

(i) $\Omega = \cup_{i_1\cdots i_K}(\Omega \cap \{T_{i_k}^k \leq t < T_{i_{k+1}}^k : 1 \leq k \leq K\}) \in \mathcal{F}_t^0$.

(ii) $\forall A_{i_1\cdots i_K} \in \mathcal{G}_{i_1\cdots i_K}$,

$$
(\cup_{i_1\cdots i_K} A_{i_1\cdots i_K} \cap \{T_{i_k}^k \leq t < T_{i_{k+1}}^k : 1 \leq k \leq K\})^c
= \cap_{i_1\cdots i_K}((\{T_{i_k}^k \leq t < T_{i_{k+1}}^k : 1 \leq k \leq K\})^c \cup (A_{i_1\cdots i_K} \cap \{T_{i_k}^k \leq t < T_{i_{k+1}}^k : 1 \leq k \leq K\}))
= \cup_{i_1\cdots i_K}(A_{i_1\cdots i_K} \cap \{T_{i_k}^k \leq t < T_{i_{k+1}}^k : 1 \leq k \leq K\})
\in \mathcal{F}_t^0.
$$

(iii) If $A_{i_1\cdots i_K}^{(l)} \in \mathcal{G}_{i_1\cdots i_K}$, then

$$
\cup_{l} \cup_{i_1\cdots i_K}(A_{i_1\cdots i_K}^{(l)} \cap \{T_{i_k}^k \leq t < T_{i_{k+1}}^k : 1 \leq k \leq K\})
= \cup_{i_1\cdots i_K}((\cup_l A_{i_1\cdots i_K}^{(l)}) \cap \{T_{i_k}^k \leq t < T_{i_{k+1}}^k : 1 \leq k \leq K\})
\in \mathcal{F}_t^0.
$$

Therefore, $\mathcal{F}_t^0$ is a $\sigma$-field.

Moreover, for any $A_{i_1\cdots i_K} \in \mathcal{G}_{i_1\cdots i_K}$,

$$
A_{i_1\cdots i_K} \cap \{T_{i_k}^k \leq t < T_{i_{k+1}}^k : 1 \leq k \leq K\}
\in \mathcal{H}_{T_{i_1}^1 \cap \cdots \cap T_{i_K}^K} \cap \{T_{i_k}^k \leq t < T_{i_{k+1}}^k : 1 \leq k \leq K\}
\subset \mathcal{H}_t,
$$

implying $\mathcal{F}_t^0 \subset \mathcal{H}_t$.

To see that $(\mathcal{F}_t^0)_{t \geq 0}$ is a filtration, for any $s < t$, let $A \in \mathcal{F}_s^0$ have the representation

$$
\cup_{i_1\cdots i_K}(A_{i_1\cdots i_K} \cap \{T_{i_{k}^k}^k \leq s < T_{i_{k+1}^k}^k : 1 \leq k \leq K\})
$$

for some $A_{i_1\cdots i_K} \in \mathcal{G}_{i_1\cdots i_K}$. Then

$$
(\cup_{i_1\cdots i_K}(A_{i_1\cdots i_K} \cap \{T_{i_{k}^k}^k \leq s < T_{i_{k+1}^k}^k : 1 \leq k \leq K\}))
= (\cup_{i_1\cdots i_K}(A_{i_1\cdots i_K} \cap \cap_{k=1}^{K} (T_{i_{k}^k}^k \leq s < T_{i_{k+1}^k}^k))) \cap \{T_{i_{k}^k}^k \leq t < T_{i_{k+1}^k}^k : 1 \leq k \leq K\}
= (\cup_{i_1\leq j_1\cdots i_K \leq j_K}(A_{i_1\cdots i_K} \cap \cap_{k=1}^{K} (T_{i_{k}^k}^k \leq s < T_{i_{k+1}^k}^k))) \{T_{i_{k}^k}^k \leq t < T_{i_{k+1}^k}^k : 1 \leq k \leq K\}
\subset \mathcal{G}_{j_1\cdots j_K} \cap \{T_{i_{k}^k}^k \leq t < T_{i_{k+1}^k}^k : 1 \leq k \leq K\}.
$$

Thus, $A \in \mathcal{F}_t^0$, implying $\mathcal{F}_s^0 \subset \mathcal{F}_t^0$.

Finally, we show that $\mathbb{P}^0$ is right continuous: let $h$ be $\cap_n \mathcal{F}_{t+h/n}^0$-measurable. Then for $\forall n \geq 1$,

$$
h = \sum_{i_1\cdots i_K} h_{i_1\cdots i_K}^{(n)} 1\{T_{i_{k}^k}^k \leq t + \frac{1}{n} < T_{i_{k+1}^k}^k : 1 \leq k \leq K\}
$$
Definition 3. Let \( X \) be a stochastic process whose state space \( E \) is a metric space \((E, \rho)\). \( X \) is genuinely stochastic, if \( \forall x \in E \) and \( t > 0 \),
(i) the law of \( X_t \) under \( P^x \) is diffuse;
(ii) \( P^x(\rho(X_t, X_0) > \epsilon) \in (0, 1) \) for any \( \epsilon > 0 \);
(iii) \( P^x(\rho(X_t, X_0) > \epsilon) \) is continuous in \( \epsilon \).

Now we can distinguish the discretely delayed filtration from its continuous counterpart.

Theorem 1. Let \( X \) be a genuinely stochastic strong Markov process, with \( \mathbb{H} = (\mathcal{H}_t)_{t \geq 0} \) being its augmented natural filtration. If \( T \) is an unbounded \( \mathbb{H} \)-stopping time, then the filtration \( \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \), which is defined by \( \mathcal{F}_t = (\mathcal{H}_0 \cap \{ t < T \}) \cup (\mathcal{H}_T \cap \{ t \geq T \}) \), is a discretely delayed filtration of \( \mathbb{H} \), but not a continuously delayed filtration of \( \mathbb{H} \).
To prove Theorem 1, it is critical to establish the following lemma and its corollary.

**Lemma 2.** Under the natural filtration $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ of a genuinely stochastic strong Markov process $X$, if $\mathcal{H}_S \subset \mathcal{H}_T$ for stopping times $S$ and $T$, then $S \leq T$ $P^x$-a.s. for all $x \in E$.

**Corollary 1.** Under the natural filtration $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ of a genuinely stochastic strong Markov process $X$, stopping times $S$ and $T$ are equal $P^x$-a.s. for all $x \in E$ if and only if $\mathcal{H}_S = \mathcal{H}_T$.

**Proof of Lemma 2.** By assumption, $S, X_S \in \mathcal{H}_T$. By the strong Markov property, for any $\epsilon > 0$ and any $x \in E$, we have

$$1_{\{S > T, \rho(X_S, X_T) > \epsilon\}} = 1_{\{S > T\}} P^x(\rho(X_S, X_T) > \epsilon | \mathcal{H}_T) = 1_{\{S > T\}} P^{X_T}(\rho(X_0, X_0) > \epsilon)_{\mid u = S - T}, \text{ } P^x - \text{a.s.}$$

On the event $\{S > T\}$, the right hand side is between 0 and 1, while the left hand side is either 0 or 1. So we must have $P^x(S > T) = 0$. This proved Lemma 2.

**Proof of Theorem 1.** Obviously, $F$ is a discretely delayed filtration of $\mathbb{H}$. Suppose $F$ is also a continuously delayed filtration of $\mathbb{H}$, with the corresponding time change $(\alpha_t)_{t \geq 0}$. We shall show $\alpha_t = T1_{\{T \leq t\}}$ and obtain a contradiction.

Fix $t > 0$ and $x \in E$. For any $A \in \mathcal{H}_{\alpha_t} = \mathcal{F}_t$, we have

$$A \cap \{t < T\} \in \mathcal{F}_t \cap \{t < T\} = \mathcal{H}_0 \cap \{t < T\}.$$

By Blumenthal’s 0-1 law, if $A \cap \{t < T\}$ is not $P^x$-negligible, then it must equal to $\{t < T\}$ $P^x$-a.s.

Now, we shall show $\alpha_t = \alpha_t1_{\{T \leq t\}}$. Assume not, then $P^x(t < T, \alpha_t > 0) > 0$. Therefore there exists $q \in (0, t)$ such that $P^x(t < T, \alpha_t > q) > 0$. Define

$$F(\epsilon) = P^x(t < T, \alpha_t > q, \rho(X_q, X_0) \leq \epsilon),$$

then $F$ is continuous in $\epsilon$, $F(0) = 0$ and $F(\infty) = P^x(t < T, \tau > q) > 0$. So there exists $\epsilon_0$, such that $0 < F(\epsilon_0) < P^x(t < T)$. But

$$\{\alpha_t > q, \rho(X_q, X_0) \leq \epsilon_0\} \in \mathcal{H}_{\alpha_t} = \mathcal{F}_t,$$

so $\{\alpha_t > q, \rho(X_q, X_0) \leq \epsilon_0, t < T\} = \{t < T\}$ $P^x$-a.s. However,

$$P^x(\alpha_t > q, \rho(X_q, X_0) \leq \epsilon_0, t < T) = F(\epsilon_0) < P^x(t < T).$$

This is a contradiction. We therefore conclude $\alpha_t = \alpha_t1_{\{T \leq t\}}$ $P^x$-a.s.. By the right continuity of $(\alpha_t)_{t \geq 0}$, for $P^x$-a.s. $\omega$, $\alpha_t(\omega) = \alpha_t(\omega)1_{\{T(\omega) \leq t\}}$, for any $t \geq 0$.

Second, it’s easy to see $\mathcal{H}_{\alpha_t} = \mathcal{F}_t \subset \mathcal{H}_T$. By Lemma 2, $\alpha_t \leq T$ $P^x$-a.s. $(\forall x \in E)$. Note $\{T \leq t\} \in \mathcal{F}_t = \mathcal{H}_{\alpha_t}$, so $(\alpha_t)_{\{T \leq t\}}$ is an $\mathbb{H}$-stopping time. Since

$$\mathcal{H}_{T\{T \leq t\}} \cap \{T > t\} = \mathcal{H}_\infty \cap \{T > t\} = \mathcal{H}_{(\alpha_t)\{T \leq t\}} \cap \{T > t\},$$

and

$$\mathcal{H}_{T\{T \leq t\}} \cap \{T \leq t\} = \mathcal{H}_T \cap \{T \leq t\} = \mathcal{F}_t \cap \{T \leq t\} = \mathcal{H}_{\alpha_t} \cap \{T \leq t\} = \mathcal{H}_{(\alpha_t)\{T \leq t\}} \cap \{T \leq t\},$$

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we have $\mathcal{H}_{t \leq t} = \mathcal{H}_{(\alpha_t)_{t \leq t}}$. By Corollary 1 and $\alpha_t = \alpha_t 1_{(T \leq t)}$, we conclude $\alpha_t = T 1_{(T \leq t)}$ $P_{\omega}$-a.s. for $(\forall x \in E)$. By the right-continuity of $(\alpha_t)_{t \geq 0}$, we conclude $\forall x \in E$, for $P_{\omega}$-a.s. $\omega$,

$$\alpha_t(\omega) = T(\omega) 1_{(T(\omega) \leq t)}$$

for $(\forall x \in E)$. By the right-continuity of $(\alpha_t)_{t \geq 0}$, we conclude $\forall x \in E$, for $P_{\omega}$-a.s. $\omega$,

$$\alpha_t(\omega) = T(\omega) 1_{(T(\omega) \leq t)}$$

for $(\forall x \in E)$. By the right-continuity of $(\alpha_t)_{t \geq 0}$, we conclude $\forall x \in E$, for $P_{\omega}$-a.s. $\omega$.

3 Delayed filtration and default intensity

Given the mathematical framework in the previous section for incomplete information, this section shows how delayed filtration induces default intensity for general Markov models: both for continuous models and for models with jumps. As special examples, we shall show the results obtained in Duffie and Lando (2001) and Collin-Dufresne et al. (2003).

Intensity of a stopping time $\tau$ Mathematically, the intensity process $(\lambda_t)_{t \geq 0}$ of a stopping time $\tau$ is associated with the compensator $A$ of $\tau$ with respect to a given filtration $\mathcal{G}$, relative to which $\tau$ is a stopping time. An increasing, right-continuous and adapted process $(A_t)_{t \geq 0}$ is called the $\mathcal{G}$-compensator of $\tau$, if $A_0 = 0$, $1_{(\tau \leq t)} - A_t$ is a $\mathcal{G}$-martingale and $A$ is $\mathcal{G}$-predictable. The intensity process $(\lambda_t)_{t \geq 0}$ of $\tau$ is then defined as the Radon–Nikodym derivative $(dA_t/dt)_{t \geq 0}$, provided that $A$ is a.s. absolutely continuous with respect to the Lebesgue measure. See Bremaud (1981).

There are a number of mathematical issues worth noting here.

First, for any non-negative random variable $\tau$ and a given filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$, $\tau$ is not necessarily an $\mathcal{F}$-stopping time. Thus, intensity process $(\lambda_t)_{t \geq 0}$ of a stopping time $\tau$ is associated with the expanded filtration $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$ of $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$, where $\tau$ is a $\mathcal{G}$-stopping time. There are more than one ways to expand $\mathcal{F}$, and we follow the simplest approach known as the minimal filtration expansion, as in Duffie and Lando (2001).

Direct computation is fairly hard except for some special cases via the route of Radon-Nikodym derivative. We adopt here the methodology known as the Meyer’s Laplacian approximation, also exploited by Duffie and Lando (2001). The essence of this approach is to calculate the default intensity $\lambda_t$ by the intuitive definition

$$\lambda_t = \lim_{h \downarrow 0} \frac{P(t + h \geq \tau > t | \mathcal{G}_t)}{h}$$

(1)

Of course, this intuitive definition of a default intensity is not the same as the mathematical definition of the Random-Nikodym derivative of the compensator of a $\mathcal{G}$-stopping time $\tau$. Under some technical conditions, however, one can show the consistency of these two notions, see Guo, Jarrow and Menn (2006). From a computation perspective, one can first compute Eq. (1), and verify it via the “Aven’s Lemma.” (See Appendix A for description of these mathematical tools.)
3.1 Continuous Markov processes

**Theorem 2.** Let $X$ be a one-dimensional, time homogeneous, continuous strong Markov process with $X_0 = x_0$. Let $\tau := \inf\{t > 0 : X_t \leq y\}$. Suppose $\tau$ has a density $f(x, y, t) dt = P^x(\tau \in dt)$, and $f(x, y, t)$ is jointly continuous in $x$, $y$ and $t$. Suppose further that the time change $(\alpha_t)_{t \geq 0}$ satisfies the following conditions: for a.s. $\omega$, $\forall T > 0$, $\exists \delta(T) = \delta(\omega, T) > 0$, such that $\forall t \in [0, T]$, 

(i) $\alpha_t(\omega) = 0$, if $t \leq \delta(\omega, T)$; 
(ii) $t - \alpha_t(\omega) \geq \delta(\omega, T)$, if $t > \delta(\omega, T)$.

Define $F = (F_t)_{t \geq 0} = (H_{\alpha_t})_{t \geq 0}$, where $H$ is the augmented natural filtration of $X$. The minimal expansion of $F$ will be $G = (G_t)_{t \geq 0}$ with $G_t = F_t \vee \sigma\{\tau \leq t : s \leq t\}$. Then, under $G$, the intensity of $\tau$ is

$$P(X_{\alpha_t}, y, t - \alpha_t) \frac{f(\tau > u)}{P^{X_{\alpha_t}}(\tau > u)} 1_{\{\tau > t\}}.$$

In particular, $\tau$ is a totally inaccessible $G$-stopping time.

Before proceeding to the proof, a few remarks are in place: condition (i) means a delay from the beginning; condition (ii) requires that the delay does not vanish during a finite amount of time.

**Proof.** Without loss of generality, we assume the probability space on which $X$ is defined is the canonical space $C[0, \infty)$ of continuous functions. So $\{\omega : \tau(\omega) > t\} = \{\omega : \tau(\omega) > \alpha_t(\omega), \tau \circ \theta_{\alpha_t}(\omega) > t - \alpha_t(\omega)\}$. By the strong Markov property, we have

$$P(\tau > t|F_t) = P(\tau > t|H_{\alpha_t}) = 1_{\{\tau > \alpha_t\}} P^{X_{\alpha_t}}(\tau > u)|_{u = t - \alpha_t},$$

and

$$P(\tau > t + h|F_t) = P(\tau > t + h|H_{\alpha_t}) = 1_{\{\tau > \alpha_t\}} P^{X_{\alpha_t}}(\tau > u)|_{u = t + h - \alpha_t}.$$

So,

$$\frac{1}{h} P(t < \tau \leq t + h|G_t) = \frac{1}{h} P(\tau < \tau \leq t + h|F_t) 1_{\{\tau > t\}}$$

$$= \frac{1}{h} P(\tau < \tau \leq t + h|F_t) 1_{\{\tau > t\}}$$

$$= \frac{1}{h} P^{X_{\alpha_t}}(u < \tau \leq u + h) \frac{f(\tau > u)}{P^{X_{\alpha_t}}(\tau > u)} 1_{\{\tau > t\}}|_{u = t - \alpha_t}$$

$$\rightarrow \lambda_t := \frac{f(X_{\alpha_t}, y, t - \alpha_t)}{P^{X_{\alpha_t}}(\tau > u)} 1_{\{\tau > t\}}|_{u = t - \alpha_t} \quad \text{as } h \downarrow 0.$$

To show $\lambda_t$ is indeed an intensity process for $\tau$, it suffices to check Aven’s conditions. And without loss of generality, we assume that $h \leq 1$, so that

$$\frac{1}{h} P(t < \tau \leq t + h|G_t) - \lambda_t$$

$$\leq \frac{1}{h} \int_{t}^{t+h} f(X_{\alpha_t}, y, s) - f(X_{\alpha_t}, y, u) ds \frac{f(\tau > u)}{P^{X_{\alpha_t}}(\tau > u)} 1_{\{\tau > t\}}|_{u = t - \alpha_t}$$

$$\leq \sup_{t - \alpha_t \leq s \leq t - \alpha_t + 1} |f(X_{\alpha_t}, y, s) - f(X_{\alpha_t}, y, t - \alpha_t)| \frac{f(\tau > u)}{P^{X_{\alpha_t}}(\tau > u)} 1_{\{\tau > t\}}|_{u = t - \alpha_t}$$

$$:= y_t.$$
We now show for a.s. \( \omega \), \( \forall T > 0 \), \( \int_0^T y_t(\omega) dt < \infty \). The measurability of \((y_t)_{t \geq 0}\) is clear as \((y_t)_{t \geq 0}\) is \( \mathbb{G} \)-adapted and right continuous, hence is optional.

Next, let \( \omega \) be such that it satisfies conditions (i)-(ii) and \( \alpha_t(\omega) \leq t \) for any \( t \geq 0 \).

For any \( t \in [0, T] \), if \( t \leq \delta(T) \), then \( \alpha_t = 0 \), and

\[
y_t \leq 2 \sup_{s \leq T+1} \frac{|f(x_0, y, s)|}{P^{x_0}(\tau > T)}.
\]

If \( t > \delta(T) \), then it is not hard to verify that

\[
y_t \leq 2 \sup_{s \leq T+1, x \in [\min_{s \leq \tau - \delta(T)} X_s, \max_{s \leq \tau - \delta(T)} X_s]} \frac{f(x, y, s)}{P^Y(\tau > T)} \quad (2)
\]

where

\[
Y(T) := \{ X_t : t \leq \tau - \delta(T), |X_t - y| = \min_{s \leq \tau - \delta(T)} |X_s - y| \}.
\]

Indeed, note \( \tau > t \Rightarrow \alpha_t \leq \tau - \delta(T) \), so on the event \( \{\tau > t\} \),

\[
P^{X_{\alpha_t}}(\tau > u) |_{u = t - \alpha_t} \geq P^Y(\tau > T), \quad (3)
\]

and

\[
\sup_{t - \alpha_t \leq s \leq t - \alpha_t + 1} |f(X_{\alpha_t}, y, s) - f(X_{\alpha_t}, y, t - \alpha_t)| \\
\leq 2 \sup_{s \leq T+1, x \in [\min_{s \leq \tau - \delta(T)} X_s, \max_{s \leq \tau - \delta(T)} X_s]} f(x, y, s).
\]

Combining the above two cases with the joint continuity of \( f \) in \((x, y, t)\), it is clear that for a.s. \( \omega \), \( y_t(\omega) \) is bounded on \([0, T]\). So \( \int_0^T y_t dt < \infty \) a.s..

In summary, \( \tau \) has an intensity process

\[
\lambda_t = \frac{f(X_{\alpha_t}, y, t - \alpha_t)}{P^{X_{\alpha_t}}(\tau > u) |_{u = t - \alpha_t}} 1_{\{\tau > t\}}.
\]

\[ \square \]

A special case of the above theorem is the following well-known result, which appeared in Collin-Dufresne et al. (2003).

**Example 3.** Suppose \( X \) is given by the Black-Scholes geometric Brownian motion model with parameters \( \mu, \sigma \). The delayed filtration \( \mathbb{F} = \sigma(W_{t_1}, \ldots, W_{t_k}) \) is expanded to \( \mathbb{G} = \sigma(W_{t_1}, \ldots, W_{t_k}) \vee \sigma(t \wedge \tau) \), then on \( \{\tau > t\} \), the default intensity of \( \mathbb{G} \)-stopping time \( \tau \) at time \( t \in [t_k, t_{k+1}) \) is

\[
- \psi_t(\frac{\mu - \frac{1}{2} \sigma^2 \alpha_t \tau}{\sigma \sqrt{t_k - t}}, \frac{\log X_{t_k} - \log X_{t}}{\sigma \sqrt{t_k - t}} - \frac{\log X_{t_k}}{\sigma}) \\
\psi_t(\frac{\mu - \frac{1}{2} \sigma^2 \alpha_t \tau}{\sigma \sqrt{t_k - t}}, \frac{\log X_{t_k} - \log X_{t}}{\sigma \sqrt{t_k - t}}).
\]

More generally, if \( X \) satisfies a one-dimensional stochastic differential equation \( dX_t = \sigma(X_t) dW_t + b(X_t) dt \) where \( b \) and \( \sigma \) satisfy some regularity conditions, then for \( \tau_y \) (the first-passage time of \( X \) hitting the level \( y \)), there is a density \( P^x(\tau_y < dt) = f(t, x, y) dt \) such that \( f(t, x, y) \) is jointly continuous (see Pauwels (1987) for details). Hence, the existence of the corresponding default intensity is implied by our result.
3.2 Diffusion process with jumps

Suppose that the asset value process $X$ follows a jump-diffusion model so that

$$X_t = X_0 e^{(\mu - \frac{1}{2} \sigma^2) t + \sigma W_t} \prod_{0 < s \leq t, \Delta \epsilon(s) \neq 0} \xi_{\epsilon(s)},$$

where $W$ is a standard one-dimensional Brownian motion, $\epsilon$ is a finite-state continuous-time Markov chain, independent of $W$ and taking values $0, 1, \cdots, S - 1$ with a known generator $(q_{ij})_{S \times S}$, and $\Delta \epsilon(s) := \epsilon(s) - \epsilon(s-)$. We set $q_i = \sum_{j \neq i} q_{ij}$, and assign to each state $i$ of $\epsilon$ ($0 \leq i \leq S - 1$) a positive random variable $\xi_i$ with distribution function $F_i$. Finally, we assume that $(\xi_i)$, $\epsilon$ and $W$ are all independent.

In particular, if $\epsilon$ is a standard Poisson process, then $X$ satisfies the SDE

$$\frac{dX_t}{X_{t-}} = \mu dt + \sigma dW_t + d\left( \sum_{i=1}^{\epsilon_t} (\xi_i - 1) \right).$$

**Theorem 3.** Define the discretely delayed filtration $\mathbb{F}$ to be the augmented minimal filtration generated by the (marked) point processes $1_{\{\tau \leq t\}}, \sum_{n=0}^{\infty} X_{1\{t_k \leq t\}},$ and $(\xi_{\epsilon(t)}, X_{T_n}, T_n)_{n \geq 1}$. Here $T_n$ is the $n$-th jump time of $\epsilon$. Take $\mathbb{G}$ to be its minimal filtration expansion. Then if $\tau > t$, $t_k \leq t < t_{k+1}$ and $T_n \leq t < T_{n+1}$, the corresponding default intensity of $\tau$ is

$$d_t = - \frac{\psi_t(\theta, t - t_k \lor T_n, \frac{1}{\sigma} \log \frac{x}{X_{t_k \lor T_n}})}{\psi(\theta, t - t_k \lor T_n, \frac{1}{\sigma} \log \frac{x}{X_{t_k \lor T_n}})} + \sum_{j \neq \epsilon(t)} q_{\epsilon(t)j} \int_0^1 F_j(dz) \phi(\theta, t - t_k \lor T_n, \frac{1}{\sigma} \log \frac{x}{X_{t_k \lor T_n}}, \frac{1}{\sigma} \log \frac{x}{X_{t_k \lor T_n}}),$$

(4)

where $\theta = \frac{\mu}{\sigma} - \frac{1}{2}$.

$$\psi(\theta, t, y) = P(\inf_{0 \leq s \leq t} W_s^{(\theta)} > y) = 1 - \int_0^t \frac{|y|}{\sqrt{2\pi s^3}} e^{-\frac{(y-\theta s)^2}{2s}} ds \text{ for } y < 0,$$

with $W_s^{(\theta)} := W_t + \theta t$, and $\psi_t$ being the derivative of $\psi$ w.r.t. the variable $t$. For $y_1 \leq y_2$,

$$\phi(\theta, t, y_1, y_2) = P(\inf_{s \leq t} W_s^{(\theta)} > y_1, W_t^{(\theta)} \leq y_2) = \Phi\left( \frac{y_2 - \theta t}{\sqrt{t}} \right) - \Phi\left( \frac{y_1 - \theta t}{\sqrt{t}} \right) - e^{2\theta y_1} \left[ \Phi\left( \frac{y_2 - 2y_1 - \theta t}{\sqrt{t}} \right) - \Phi\left( \frac{-y_1 - \theta t}{\sqrt{t}} \right) \right],$$

with $\Phi(x)$ being the distribution of a standard normal random variable.

3.3 Regime switching process

In a regime-switching model, the asset value process $X$ is assumed to follow a diffusion process given by

$$dX_t = X_t \mu_{\epsilon(t)} dt + \sigma_{\epsilon(t)} X_t dW_t,$$
where $W$ is a standard one-dimensional Brownian motion, and $\epsilon$ is a finite-state continuous-time Markov chain, independent of $W$ and taking values $0, 1, \cdots, S - 1$ with a known generator $(q_{ij})_{S \times S}$. Finally, the drift and volatility coefficients, $\mu(\cdot)$ and $\sigma(\cdot)$, are functions of $\epsilon$.

We consider the discretely delayed filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ generated by the point process $\sum_{k=1}^{\infty} X_{t_k} 1_{\{t \leq t_k\}}$ and the marked point process $(\epsilon(T_n), X_{T_n}, T_n)_{n \geq 1}$.

To this end, let us start with a general and multi-dimensional Markov setting. Let $X = (X_t)_{t \geq 0}$ be a Markov process under a risk-neutral measure $Q$, with a general state space $E$. Let $\mathcal{H} = (\mathcal{H}_t)_{t \geq 0}$ be a subset of $E$ and $\tau := \inf\{t > 0 : X_t \in U\}$. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the natural filtration of $X$. Suppose the maturity date of the bond is $T$, then it’s easy to see from the Markov property that

$$
\psi_t(\theta(t), t - t_k \lor T_n, \frac{1}{\sigma_\epsilon(t)} \log \frac{x}{X_{t_k \lor T_n}})

\frac{\psi(\theta(t), t - t_k \lor T_n, \frac{1}{\sigma_\epsilon(t)} \log \frac{x}{X_{t_k \lor T_n}})}{\psi(\theta(t), t - t_k \lor T_n, \frac{1}{\sigma_\epsilon(t)} \log \frac{x}{X_{t_k \lor T_n}})}
$$

where

$$
\psi(\theta(t), t - t_k \lor T_n, \frac{1}{\sigma_\epsilon(t)} \log \frac{x}{X_{t_k \lor T_n}}) = P\left( \inf_{0 \leq s \leq t} W_s(\theta(t)) > y \right) = 1 - \int_0^t \frac{|y|}{\sqrt{2\pi s^3}} e^{-\frac{(y - \theta(t))^2}{2s}} ds \; \text{ for } y < 0,
$$

with $W_t(\theta) := W_t + \theta t$, and $\psi_t$ being the derivative of $\psi$ w.r.t. the variable $t$.

4 Risky Debt Pricing: relation between complete information and delayed information

Consider a zero-coupon bond issued by the firm paying $\$1$ at time $T$ if there is no default, and $\$R$ at time $T$ if the firm defaults prior to time $T$. Here $R(t, T) \in [0, 1]$ is called the face value of debt recovery rate process (see Jarrow and Turnbull (1995)). Similar results hold for other recovery rate processes (see Bielecki and Rutkowski (2002) for the relevant alternatives). For simplicity, we assume that the interest rate process is deterministic.

Now we investigate the relation between values of defaultable zero recovery rate, zero-coupon bond, under the two different filtration structures, complete and delayed.

To this end, let us start with a general and multi-dimensional Markov setting. Let $X = (X_t)_{t \geq 0}$ be a Markov process under a risk-neutral measure $Q$, with a general state space $E$. Let $\mathcal{H} = (\mathcal{H}_t)_{t \geq 0}$ be a subset of $E$ and $\tau := \inf\{t > 0 : X_t \in U\}$. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the natural filtration of $X$. Suppose the maturity date of the bond is $T$, then it’s easy to see from the Markov property that

$$
G_t := \sigma(X_{t_1}, \cdots, X_{t_k}) \lor \sigma(t \land \tau)
$$

for $t_k \leq t < t_{k+1}$, we have

$$
V^B(t, T) = 1_{\{\tau > t\}} \int_0^T \frac{V_A(t, T)}{V_A(t_k, t)} dt.
$$

Here $V_A(t_k, T) = Q(\tau > T | \mathcal{H}_t)$ and $V^B(t_k, T) = Q(\tau > T | G_t)$.

---

3Here, $X_t$ corresponds to the asset value process, plus any additional processes needed for the interest rate and recovery rate process.
Proof. Suppose $t \in [t_k, t_{k+1})$, then

\[
V^B(t, T) = 1_{\{\tau > t\}} \frac{Q(\tau > T| X_{t_1}, \ldots, X_{t_k})}{Q(\tau > t| X_{t_1}, \ldots, X_{t_k})} \\
= 1_{\{\tau > t\}} \frac{Q(1_{\{\tau > t_k\}} Q(X_s \notin U, \forall s \in [t_k, T]| X_{t_1}, \ldots, X_{t_k})}{Q(1_{\{\tau > t\}} Q(X_s \notin U, \forall s \in [t_k, t]| X_{t_1}, \ldots, X_{t_k})} \\
= 1_{\{\tau > t\}} \frac{Q(X_s \notin U, \forall s \in [t_k, T]| X_{t_k})}{Q(X_s \notin U, \forall s \in [t_k, t]| X_{t_k})} V^A(t_k, T) = 1_{\{\tau > t\}} V^A(t_k, T).
\]

Note first that investors $\mathcal{A}$ and $\mathcal{B}$ have different valuations, except at time $t_i$ (0 ≤ $i$ ≤ $k$). That is, $V^A(t_i, T) = V^B(t_i, T)$, because at time $t_i$ both investors have the same information. Secondly, we do not necessarily have $V^A(t, T) > V^B(t, T)$ or $V^A(t, T) < V^B(t, T)$. Indeed, when $t_k < t$, the price for investor $\mathcal{B}$ is based on the “dated” information $X_{t_k}$ as compared to that of investor $\mathcal{A}$. This dated information could generate a higher or lower price, depending upon the complete information known only to investor $\mathcal{A}$. Finally, as $t \downarrow t_k$, $V^B(t, T) \to V^A(t, T)$, i.e., as investor $\mathcal{B}$’s information is updated, his price again agrees with investor $\mathcal{A}$. This relationship is independent of the risk-neutral measure under consideration.

References


4.1 Appendix A

**Theorem 5. (Meyer’s Laplacian Approximation)** Let $Z$ be a càdlàg positive supermartingale of the class $(D)$ (i.e., the set $\{Z_T: T$ is a stopping time $\tau$ satisfying $T < \infty\}$ is uniformly integrable), with $\lim_{t \to \infty} E\{Z_t\} = 0$. Let $Z = M - A$ be its Doob-Meyer decomposition. Define

$$A^h_t = \int_0^t Z_s - E\{Z_{s+h}|\mathcal{F}_s\} \frac{ds}{h}.$$

Then for any stopping time $T$, $A^h_T = \lim_{h \to 0} A^h_T$, in the sense of the weak topology $\sigma(L^1, L^\infty)$.

**Proof.** See Meyer (1966), page 119, T29. \qed
Lemma 3. (T. Aven (1985)) Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) \) be a filtered probability space that satisfies usual hypothesis. Let \( (N_t)_{t \geq 0} \) be a counting process. Assume that \( E\{N_t\} < \infty \) for all \( t \). Let \( \{h_n\}_{n \geq 1} \) be a sequence which decreases to zero and let for each \( n \) \( (Y^n_n)_{t \geq 0} \) be a measurable version of the process \( (E\{N_{t+h_n} - N_t|\mathcal{F}_t\}/h_n)_{t \geq 0} \). Assume that the following statements hold with \( (\lambda_t)_{t \geq 0} \) and \( (y_t)_{t \geq 0} \) being non-negative measurable processes:

(i) for each \( t \), \( \lim_n Y^n_t = \lambda_t \) a.s.;

(ii) for each \( t \), there exists for almost all \( \omega \) an \( n_0 = n_0(t, \omega) \) such that

\[
|Y^n_t(\omega) - \lambda_t(\omega)| \leq y_n(\omega), \quad s \leq t, n \geq n_0
\]

(iii) \( \int_0^t y_s ds < \infty \), a.s. \( 0 \leq t < \infty \).

Then \( N_t - \int_0^t \lambda_s ds \) is an \( \mathcal{F}_t \)-martingale, i.e., \( (\int_0^t \lambda_s ds)_{t \geq 0} \) is the compensator of \( (N_t)_{t \geq 0} \).

4.2 Appendix B: Proof of Theorem 4

Proof. For ease of exposition, let us denote by \( \mathcal{F}_{k,n} \) the \( \sigma \)-field generated by \( W_{t_1}, \ldots, W_{t_k}, T_1, \ldots, T_n, \epsilon(T_1), \ldots, \epsilon(T_n), \) and \( W_{T_1}, \ldots, W_{T_n} \). We further denote \( P(\cdot|\mathcal{F}(y_k, u_n, x_n, v_n)) \) for \( P(\cdot|W_{t_1} = y_1, \ldots, W_{t_k} = y_k, T_1 = u_1, \ldots, T_n = u_n, W_{T_1} = x_1, \ldots, W_{T_n} = x_n, \epsilon(T_1) = v_1, \ldots, \epsilon(T_n) = v_n) \).

Note on the event \( \{\tau > t, T_n \leq t < T_{n+1}\} \), when \( T_1, \ldots, T_n \) and \( \epsilon(T_1), \ldots, \epsilon(T_n) \) are known, there is a one-to-one correspondence between \( (W_{t_1}, \ldots, W_{t_k}, W_{T_1}, \ldots, W_{T_n}) \) and \( (X_{t_1}, \ldots, X_{t_k}, X_{T_1}, \ldots, X_{T_n}) \). We denote by \( x^i_{t_k} \) the value of \( X_{T_i} \) \((1 \leq i \leq n)\) and by \( y^j_{t_k} \) the value of \( X_{t_j} \) \((1 \leq j \leq k)\) given \( \mathcal{F}(y_k, u_n, x_n, s_n) \). Finally, we let \( \theta_i = \frac{\mu_i}{\sigma^2} - \frac{\sigma^2}{2} \) \((0 \leq i \leq S - 1)\), \( X_i = \exp\{(\mu_i - \frac{\sigma^2}{2})t + \sigma_i W_t\} \)(\(0 \leq i \leq S - 1\)), and \( (\mathcal{F}_{W}^n)_{t \geq 0} \) be the natural filtration of \( W \).

Note for \( t \in [t_k, t_{k+1}) \), one has on the event \( \{\tau > t, T_{n+1} > t \geq T_n\} \)

\[
P(t + h \geq \tau > t|\mathcal{G}_t) = P(\tau > t + h|\mathcal{G}_t) = 1 - P(\tau > t + h, T_{n+1} > t \geq T_n|\mathcal{F}_{k,n})
\]

by the Bayes’ formula and the fact that \( \mathcal{G}_t \cap \{\tau > t, T_n \leq t < T_{n+1}\} = \mathcal{F}_{k,n} \cap \{\tau > t, T_n \leq t < T_{n+1}\} \).

Since \( W \) is independent of \( \epsilon \), by conditioning on \( \mathcal{F}_{W}^{t_k} \) or \( \mathcal{F}_{W}^{u_n} \) and by the strong Markov property, we have
\begin{align*}
P(\tau > t, T_{n+1} > t \geq T_n | \mathcal{F}(y_k, u_n, x_n, v_n))
= P\left(x'_i \inf_{u_i \leq s < u_{i+1}} \frac{X_s^{(v_i)}}{X_{u_i}^{(v_i)}} > x, i \leq n - 1, x'_n \inf_{u_n \leq s \leq t} \frac{X_s^{(v_n)}}{X_{u_n}^{(v_n)}} > x | \mathcal{F}(y_k, u_n, x_n, v_n)\right) \\
\times P(T_{n+1} > t | T_n = u_n, \epsilon(T_n) = v_n)
= \begin{cases} 
\psi(\theta_{v_n}, t - u_n, \frac{1}{\sigma_{v_n}} \log \frac{x}{x_n}) e^{-q_{v_n}(t-u_n)} & \text{if } u_n \geq t_k, \ P\left(x'_i \inf_{u_i \leq s < u_{i+1}} \frac{X_s^{(v_i)}}{X_{u_i}^{(v_i)}} > x, i \leq n - 1, x'_n \inf_{u_n \leq s \leq t} \frac{X_s^{(v_n)}}{X_{u_n}^{(v_n)}} > x | \mathcal{F}(y_k, u_n, x_n, v_n)\right) \\
\psi(\theta_{v_n}, t - t_k, \frac{1}{\sigma_{v_n}} \log \frac{x}{y_k}) e^{-q_{v_n}(t-u_n)} & \text{if } u_n < t_k, \ P\left(x'_i \inf_{u_i \leq s < u_{i+1}} \frac{X_s^{(v_i)}}{X_{u_i}^{(v_i)}} > x, i \leq n - 1, x'_n \inf_{u_n \leq s \leq t_k} \frac{X_s^{(v_n)}}{X_{u_n}^{(v_n)}} > x | \mathcal{F}(y_k, u_n, x_n, v_n)\right).
\end{cases}
\end{align*}

By splitting \(\{T_{n+1} > t \geq T_n\}\) into \(\{T_{n+1} > t + h > t \geq T_n\}\), \(\{T_{n+2} > t + h \geq T_{n+1} > t \geq T_n\}\) and \(\{t + h \geq T_{n+2} > T_{n+1} > t \geq T_n\}\), we get

\[
P(\tau > t + h, T_{n+1} > t \geq T_n | \mathcal{F}_{k,T_n,W_T_n})
= e^{-q_{v_n}(T_n+h)} \psi(\theta_{v_n}, t + h - t_k \vee T_n, \frac{1}{\sigma_{v_n}} \log \frac{x}{x_{T_n+W_T_n}}) + I + II,
\]

where

\[
I = \frac{P(\tau > t + h, T_{n+2} > t + h \geq T_{n+1} > t \geq T_n | \mathcal{F}_{k,n})}{P(\tau > t, T_{n+1} > t \geq T_n | \mathcal{F}_{k,n})},
\]

\[
II = \frac{P(\tau > t + h, t + h \geq T_{n+2} > T_{n+1} > t \geq T_n | \mathcal{F}_{k,n})}{P(\tau > t, T_{n+1} > t \geq T_n | \mathcal{F}_{k,n})}.
\]

The calculation of I is similar to the previous one, except that we need to proceed the calculation...
conditioned on \( T_{n+1} \) and \( \epsilon(T_{n+1}) \):

\[
P(r > t + h, T_{n+2} > t + h \geq T_{n+1} > t \geq T_n) = \frac{X_s^{(v_{n+1})}}{X_{u_j}^{(v_{n+1})}} > x, 0 \leq i \leq n, \quad T_{n+2} > t + h | F(u_k, u_n, x_n, v_n), T_{n+1} = u_{n+1}, \epsilon(T_{n+1}) = v_{n+1}) \times P(T_{n+1} \in du_{n+1}, \epsilon(T_{n+1}) = v_{n+1} | F(y_k, u_n, x_n, v_n)) \]

\[
\begin{cases}
\text{if } u_n \geq t_k, \quad P(x'_i, \inf_{u_i \leq s \leq u_{n+1}} X_s^{(v_i)} \bigg| X_{u_j}^{(v_i)} > x, i \leq n - 1 | F(y_k, u_n, x_n, v_n)) \\
\sum_{v_{n+1} \neq v_n} \int_t^{t+h} q_{v_n v_{n+1}} e^{-q_{v_n}(t+h-u_{n+1})} q_{v_n} e^{-q_{v_n}(u_{n+1}-u_n)} \\
E[\psi(\theta_{v_{n+1}}, t + h - u_{n+1}, \frac{1}{\sigma_{v_{n+1}}} \log \frac{x}{x_n} - \frac{\sigma_{v_n}}{\sigma_{v_{n+1}}} W^{(\theta_{v_n})}_{u_{n+1} - u_n})] \\
\inf_{0 \leq s \leq u_{n+1} - u_n} W_s^{(\theta_{v_n})} > \frac{1}{\sigma_{v_n}} \log \frac{x}{y_k} du_{n+1} \\
\text{if } u_n < t_k, \quad P(x'_i, \inf_{u_i \leq s \leq u_{n+1}} X_s^{(v_i)} \bigg| X_{u_j}^{(v_i)} > x, i \leq n - 1, \quad x'_n \inf_{u_n \leq s \leq t_k} X_s^{(v_n)} > x \\
|F(y_k, u_n, x_n, v_n)) \sum_{v_{n+1} \neq v_n} \int_t^{t+h} q_{v_n v_{n+1}} e^{-q_{v_n}(t+h-u_{n+1})} q_{v_n} e^{-q_{v_n}(u_{n+1}-u_n)} \\
E[\psi(\theta_{v_{n+1}}, t + h - u_{n+1}, \frac{1}{\sigma_{v_{n+1}}} \log \frac{x}{y_k} - \frac{\sigma_{v_n}}{\sigma_{v_{n+1}}} W^{(\theta_{v_n})}_{u_{n+1} - t_k})] \\
\inf_{0 \leq s \leq u_{n+1} - t_k} W_s^{(\theta_{v_n})} > \frac{1}{\sigma_{v_n}} \log \frac{x}{y_k} du_{n+1}.
\end{cases}
\]

Therefore \( I \) is

\[
\sum_{j \neq \epsilon(T_n)} \int_t^{t+h} q_{\epsilon(T_n) j} e^{-q_{\epsilon(T_n)}(t+h-u_{n+1})} e^{-q_{\epsilon(T_n)}(u_{n+1}-T_n)} \psi(\theta_{\epsilon(T_n)}), u_{n+1} - t_k \lor T_n, \frac{\log \frac{x}{\sigma_{\epsilon(T_n)}}}{\sigma_{\epsilon(T_n)}} du_{n+1}
\]

\[
\psi(\theta_{\epsilon(T_n)}, t - t_k \lor T_n, \frac{1}{\sigma_{\epsilon(T_n)}} \log \frac{x}{X_{t_k \lor T_n}})
\]

As \( h \downarrow 0 \), the first term tends to \( \sum_{j \neq \epsilon(T_n)} q_{\epsilon(T_n) j} = q_{\epsilon(T_n)} \). For the second term, notice

\[
(*) \leq \int_0^h \sum_{j \neq v_n} q_{v_n, j} E[1 - \psi(\theta_j, h - u, \frac{1}{\sigma_j} \log \frac{x}{z} - \frac{\sigma_{v_n}}{\sigma_j} W^{(\theta_{v_n})}_{u + (t_k - \alpha)}]; \log \frac{x}{z} < \sigma_{v_n} W^{(\theta_{v_n})}_{u + (t_k - \alpha)}] du,
\]

where \( v_n = \epsilon(T_n), z = X_{t_k \lor T_n}, \) and \( \alpha = t_k \lor T_n \). By the bounded convergence theorem, the expectation is a continuous function of \( u \), so \( \lim_{h \downarrow 0} (*) = 0 \). Moreover, it is clear that \( (*) \leq q_{\epsilon(T_n)} h \).
Similarly, to calculate $II$, recall

$$P(\tau > t + h, t + h \geq T_{n+2} > T_{n+1} > t \geq T_n|\mathcal{F}(y_k, u_n, x_n, v_n)) \leq P(x'_i \inf_{u_i \leq s \leq u_{i+1}} X_{u_i}^{(v_i)} > x, i \leq n - 1, x'_n \inf_{u_n \leq s \leq t} X_{u_n}^{(v_n)} > x|\mathcal{F}(t_k, u_n, x_n, v_n))Ch^2,$$

for some constant $C$. Therefore $II \leq Ch^2$.

In summary, the above calculations show that, on the event $\{\tau > t, T_n \leq t < T_{n+1}, t_k \leq t < t_{k+1}\}$,

$$\lim_{h \downarrow 0} \frac{1}{h} P(t + h \geq \tau > t|\mathcal{F}_t) = -\frac{\psi_t(\theta(T_n), t - t_k \vee T_n, \frac{1}{\sigma(T_n)} \log \frac{x}{X_{t_k \vee T_n}})}{\psi(\theta(T_n), t - t_k \vee T_n, \frac{1}{\sigma(T_n)} \log \frac{x}{X_{t_k \vee T_n}})} \cdot (I - II) \frac{1}{h} + |d_t| + \frac{I}{h} + Ch$$

Finally, we check that Aven’s lemma holds: when $\tau > t$, $T_n \leq t < T_{n+1}$ and $t_k \leq t < t_{k+1}$, for $h \leq 1$,

$$\left| \frac{1}{h} P(t + h \geq \tau > t|\mathcal{F}_t) - d_t \right|$$

$$\leq \left| \frac{1}{h} \left[ \left( 1 - e^{-q(T_n)}h \frac{\psi(\theta(T_n), t + h - t_k \vee T_n, \frac{1}{\sigma(T_n)} \log \frac{x}{X_{t_k \vee T_n}})}{\psi(\theta(T_n), t - t_k \vee T_n, \frac{1}{\sigma(T_n)} \log \frac{x}{X_{t_k \vee T_n}})} - I - II \right) \frac{1}{h} + |d_t| + \frac{I}{h} + Ch \right|$$

$$\leq \left( \frac{1}{h} \left[ \frac{\int_{t+h}^{t} \psi(\theta(T_n), s - t_k \vee T_n, \frac{1}{\sigma(T_n)} \log \frac{x}{X_{t_k \vee T_n}}) ds}{h \psi(\theta(T_n), t - t_k \vee T_n, \frac{1}{\sigma(T_n)} \log \frac{x}{X_{t_k \vee T_n}})} + |d_t| \right. \right.$$