Stochastic Games and Mean Field Games with Singular Controls

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Abstract

In this paper we show that a class of N-player stochastic games with singular controls of finite variation can be approximated by the corresponding MFGs with singular controls of bounded velocity. This follows from three results, each of independent interest. The first establishes the convergence of singular control problems of bounded velocity to singular control problems of finite variation; the second shows that an N-player stochastic game with singular controls of bounded velocity can be approximated by the corresponding MFG with singular controls of bounded velocity; the third shows the existence and uniqueness of the optimal control to the MFG of bounded velocity.

1 Introduction

N-player non-zero-sum stochastic games are notoriously hard to analyze. Recently, the theory of Mean Field Games (MFGs), pioneered by Lasry and Lions (2007) and Huang, Malhamé, and Caines (2006), presents a powerful approach to study stochastic games of a large population with small interactions. MFG avoids directly analyzing the difficult N-player stochastic games. Instead, it approximates the dynamics and the objective function under the notion of population's probability distribution flows, a.k.a., mean information processes. Under proper technical conditions, the optimal control to an MFG is shown to be an ϵ-*Nash Equilibrium* (ϵ-NE) to its corresponding N-player game. As such, MFGs provide an elegant and analytically feasible framework to approximate N-player stochastic games.

The literature on MFGs is expanding rapidly. In addition to the PDEs approach (see also Guéant, Lasry, and Lions (2010)), there are approaches based on backward stochastic differential equations (BSDEs) by Buckdahn, Djehiche, Li, and Peng (2009) and Buckdahn, Li, and Peng (2009), the probabilistic approach by Carmona and Delarue (2013, 2014) and Carmona and Lacker (2015), and the dynamic programming method on McKean–Vlasov controls by Pham and Wei (2015, 2016). Recently, Fischer (2014) connects symmetric N-player games with MFGs, Lacker (2015) analyzes MFGs by formulating controlled martingale problems, Carmona, Delarue, and Lacker (2016) and Nutz (2016) add common noise to MFGs, Carmona and Zhu (2014) and Sen and Caines (2017) add major and minor players to MFGs, and Gomes, Patrizi, and Voskanyan (2013) and Cardaliaguet and Lehalle (2016) study stationary MFGs. All these theoretical developments are within the framework of regular controls where controls are absolutely continuous.

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Compared to regular controls, singular controls provide a more general mathematical framework where controls are allowed to be discontinuous. Though more natural for practical engineering and economics problems, singular controls are in general more challenging to analyze. For instance, analyzing singular control problems amounts to adding possibly state-dependent gradient constraints to the underlying Hamilton–Jacobi–Bellman (HJB) equations. Moreover, the Hamiltonian for singular controls diverges and the standard stochastic maximal principle fails. To overcome these technical difficulties for MFGs with singular controls, Zhang (2012) and Hu, Øksendal, and Sulem (2014) propose and establish relaxed stochastic maximal principles to prove the existence of optimal controls to MFGs, while Fu and Horst (2016) adopt the notion of relaxed controls to prove the uniqueness and existence of solution to MFGs in a finite-time horizon.

Our work. There are two types of singular controls, namely, single controls of finite variation and singular controls of bounded velocity. Unlike singular controls of finite variation, singular controls of bounded velocity share some nice properties with regular controls and are easier to analyze. In this paper we show that N-player stochastic games with singular controls of finite variation can in fact be approximated under the NE criterion by MFGs with singular controls of bounded velocity. The key idea is to introduce an intermediate stochastic game: an N-player game with singular controls of bounded velocity.

To find the relationships among these games of different types, we first analyze the relationship between the underlying singular control problems. Assuming $T < \infty$, we show that under proper assumptions, the value function of singular controls of bounded velocity converges to that of singular controls of finite variation as the bound θ goes to infinity (Theorem 1). Note that a similar result was established in a different problem setting in Hernández-Hernández, Pérez, and Yamazaki (2016). We then establish the existence, uniqueness, and the smoothness for the solution to the MFG with singular controls of bounded velocity (Theorem 2). Combined, we next show that (i) the optimal control to the MFG with singular controls of bounded velocity is an ϵ_N-NE to an N-player game with singular controls of bounded velocity with $\epsilon_N = O(\frac{1}{\sqrt{N}})$, and (ii) it is an $(\epsilon_N + \epsilon_\theta)$-NE to an N-player game with singular controls of finite variation, where ϵ_θ is an error term that depends on θ (Theorem 3).

2 Problem formulations and main results

Throughout the paper, we will use the following notation, unless otherwise specified.

Notation.

- $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le \infty}, P)$ is a probability space and $W^i = \{W_t^i\}_{0 \le t \le \infty}$ are i.i.d. standard Brownian motions in this space for $i = 1, \dots, N < \infty$.
- $\mathcal{P}(\mathbb{R})$ is the set of all probability measures on $\mathbb{R}$.
- $\mathcal{P}_p(\mathbb{R})$ is the set of all probability measures of pth order on $\mathbb{R}$. That is

$$\mathcal{P}_p(\mathbb{R}) = \left\{ \mu \in \mathcal{P}(\mathbb{R}) \middle| \left(\int_{\mathbb{R}} |x|^p \mu(dx) \right)^{\frac{1}{p}} < \infty \right\}.$$

- The pth order Wasserstein metric on $\mathcal{P}_p(\mathbb{R})$ is defined as

$$D^p(\mu, \mu') = \inf_{\tilde{\mu}} \left(\int_{\mathbb{R} \times \mathbb{R}} |y - y'|^p \tilde{\mu}(dy, dy') \right)^{\frac{1}{p}},$$

where μ, μ' are any probability measures in $\mathcal{P}_p(\mathbb{R})$ and $\tilde{\mu}$ is a coupling of μ and μ'.

- $\mathcal{M}_{[0,T]} \subset \mathcal{C}([0,T] : \mathcal{P}_1(\mathbb{R}))$ is a class of flows of probability measures $\{\mu_t\}_{0\le t\le T}$ on $[0,T]$. That is, there exists a positive constant c such that

$$\mathcal{M}_{[0,T]} = \left\{ \{\mu_t\}_{0\le t\le T} \in \mathcal{C}([0,T] : \mathcal{P}_1(\mathbb{R})) \,\middle|\, \sup_{s\neq t} \frac{D^1(\mu_t,\mu_s)}{|t-s|^{\frac{1}{2}}} \le c, \sup_{t\in[0,T]} \int_{\mathbb{R}} |x|^2 \mu_t(dx) \le c \right\},$$

where $\mathcal{M}_{[0,T]}$ is a metric space endowed with the metric

$$d_{\mathcal{M}}\left(\{\mu_t\}_{0\le t\le T}, \{\mu'_t\}_{0\le t\le T} \right) = \sup_{0\le t\le T} D^1(\mu_t, \mu'_t). \tag{1}$$

- $\mathcal{M}_{[0,\infty)} \subset \mathcal{C}([0,\infty) : \mathcal{P}_1(\mathbb{R}))$ is a class of flows of probability measures $\{\mu_t\}_{0\le t<\infty}$ on $[0,\infty)$. That is, there exists a positive constant c_T which depends on T such that

$$\mathcal{M}_{[0,\infty)} = \Bigg\{ \{\mu_t\}_{0\le t<\infty} \in \mathcal{C}([0,\infty) : \mathcal{P}_1(\mathbb{R})) \,\Bigg|\, \sup_{s\neq t} \frac{D^1(\mu_t,\mu_s)}{|t-s|^{\frac{1}{2}}} \le c, \text{ and }$$
$$\text{for each } T \in [0,\infty), \sup_{t\in[0,T]} \int_{\mathbb{R}} |x|^2 \mu_t(dx) \le c_T \Bigg\},$$

where $\mathcal{M}_{[0,\infty)}$ is a metric space endowed with the metric

$$d^{\beta}_{\mathcal{M}}\left(\{\mu_t\}_{0\le t<\infty}, \{\mu'_t\}_{0\le t<\infty} \right) = \int_0^\infty e^{-\beta t} D^1(\mu_t, \mu'_t) dt, \tag{2}$$

for some sufficient large $\beta > 0$.

- $Lip(\psi)$ is a Lipschitz coefficient of ψ for any given Lipschitz function ψ. That is, $|\psi(x) - \psi(y)| \le Lip(\psi)|x-y|$ for any $x, y \in \mathbb{R}$.

- $\mathcal{L}\psi(x) = b(x)\partial_x\psi(x) + \frac{1}{2}\sigma^2(x)\partial_{xx}\psi(x)$ is the infinitesimal generator for any stochastic process $dx_t = b(x_t)dt + \sigma(x_t)dW_t$ and any $\psi(x) \in \mathcal{C}^2$.

- A function f is said to be of a polynomial growth if $|f(x)| \le c(|x|^k + 1)$ for some constants c and k, for all x.

2.1 N-player stochastic games

N-player game with singular controls of a finite variation. Fix a time $T < \infty$ and suppose there are N identical players in the game. Denote $\{x^i_t\}_{s\le t\le T}$ as the state process in $\mathbb{R}$ for player i ($i = 1, \dots, N$), with $x^i_{s-} = x^i$ starting from time $s \in [0,T]$. Now assume that the dynamics of $\{x^i_t\}$ follows, for $s \le t \le T$,

$$dx^i_t = \frac{1}{N}\sum_{j=1}^N b_0(x^i_t, x^j_t)dt + \sigma dW^i_t + d\xi^{i+}_t - d\xi^{i-}_t, \quad x^i_{s-} = x^i, \tag{3}$$

where $b_0 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and σ is a positive constant. Here $\xi^i_\cdot = (\xi^{i+}_\cdot, \xi^{i-}_\cdot)$ is the control by player i with $(\xi^{i+}_\cdot, \xi^{i-}_\cdot)$ nondecreasing, càdlàg, $\xi^{i+}_{s-} = \xi^{i-}_{s-} = 0$, $\mathbb{E}\left[\int_s^T d\xi^{i+}_t\right] < \infty$, and $\mathbb{E}\left[\int_s^T d\xi^{i-}_t\right] < \infty$.

Given Eqn. (3), the objective of player i is to minimize, over an appropriate control set $\mathcal{U}^N_{T,\infty}$, her cost function $J^{i,N}_{T,\infty}(s, x^i, \xi^{i+}_\cdot, \xi^{i-}_\cdot; \xi^{-i}_\cdot)$. That is

$$\inf_{(\xi^{i+}_\cdot,\xi^{i-}_\cdot)\in\mathcal{U}^N_{T,\infty}} J^{i,N}_{T,\infty}(s, x^i, \xi^{i+}_\cdot, \xi^{i-}_\cdot; \xi^{-i}_\cdot) = \inf_{(\xi^{i+}_\cdot,\xi^{i-}_\cdot)\in\mathcal{U}^N_{T,\infty}} \mathbb{E}\left[\int_s^T \frac{1}{N}\sum_{j=1}^N f_0(x^i_t, x^j_t)dt + \gamma_1 d\xi^{i+}_t + \gamma_2 d\xi^{i-}_t\right]. \tag{N-FV}$$

Here $\xi^{-i}_\cdot = \{(\xi^{j+}_\cdot, \xi^{j-}_\cdot)\}^n_{j=1,j\neq i}$ is the set of controls for all the players except for player i, the cost function $f_0 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, γ_1 and γ_2 are constants, and

$$\mathcal{U}^N_{T,\infty} = \left\{(\xi^+_\cdot, \xi^-_\cdot) \;\middle|\; \xi^+_t \text{ and } \xi^-_t \text{ are } \mathcal{F}^{(x^1_t,\ldots,x^N_t)}_t\text{-adapted, cádlág, nondecreasing,} \right.$$
$$\left. \xi^+_{s-} = \xi^-_{s-} = 0, \mathbb{E}\left[\int_s^T d\xi^+_t\right] < \infty, \text{ and } \mathbb{E}\left[\int_s^T d\xi^-_t\right] < \infty, \text{ for } 0 \le s \le t \le T\right\},$$

with $\{\mathcal{F}^{(x^1_t,\ldots,x^N_t)}_t\}_{s\le t\le T}$ the natural filtration of $\{x^1_t, \ldots, x^N_t\}_{s\le t\le T}$.

Note that in (N-FV), both the drift term in the dynamics and the objective function are affected by the state of the player i, and the states of all the other players. In general, this type of stochastic game is difficult to analyze.

N-player game with singular controls of bounded velocity. If one restricts the controls $(\xi^{i+}_\cdot, \xi^{i-}_\cdot)$ to be with a bounded velocity such that

$$d\xi^{i+}_t = \dot{\xi}^{i+}_t dt, \quad d\xi^{i-}_t = \dot{\xi}^{i-}_t dt,$$

with $0 \le \dot{\xi}^{i+}_t, \dot{\xi}^{i-}_t \le \theta$ for a constant $\theta > 0$. Then game (N-FV) becomes

$$\inf_{(\xi^{i+}_\cdot,\xi^{i-}_\cdot)\in\mathcal{U}^N_{T,\theta}} J^{i,N}_{T,\theta}(s, x^i, \xi^{i+}_\cdot, \xi^{i-}_\cdot; \xi^{-i}_\cdot) = \inf_{(\xi^{i+}_\cdot,\xi^{i-}_\cdot)\in\mathcal{U}^N_{T,\theta}} \mathbb{E}\left[\int_s^T \frac{1}{N}\sum_{j=1}^N f_0(x^i_t, x^j_t)dt + \gamma_1 \dot{\xi}^{i+}_t dt + \gamma_2 \dot{\xi}^{i-}_t dt\right],$$
$$\text{subject to} \quad dx^i_t = \frac{1}{N}\sum_{j=1}^N b_0(x^i_t, x^j_t)dt + \sigma dW^i_t + \dot{\xi}^{i+}_t dt - \dot{\xi}^{i-}_t dt, \quad x^i_s = x^i. \tag{N-BD}$$

Here the admissible set is given by

$$\mathcal{U}^N_{T,\theta} = \left\{(\xi^+_\cdot, \xi^-_\cdot) \;\middle|\; (\xi^+_\cdot, \xi^-_\cdot) \in \mathcal{U}^N_{T,\infty}, 0 \le \dot{\xi}^+_t, \dot{\xi}^-_t \le \theta, \text{ for } 0 \le s \le t \le T\right\}.$$

2.2 Corresponding MFG games when $N \to \infty$

Game (N-BD) is easier to analyze than game (N-FV). Indeed, assume that all N-players are identical. That is, for each time $t \in [0, T]$, all x^i_t have the same probability distribution. Define $\varepsilon_t = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \delta_{x^i_t}$ as the empirical distribution of x^i_t. Then, according to SLLN, as $N \to \infty$,

$$\frac{1}{N}\sum_{j=1}^N b_0(x_t, x^j_t) \to \int_{\mathbb{R}} b_0(x_t, y)\varepsilon_t(dy) = b(x_t, \varepsilon_t),$$
$$\frac{1}{N}\sum_{j=1}^N f_0(x_t, x^j_t) \to \int_{\mathbb{R}} f_0(x_t, y)\varepsilon_t(dy) = f(x_t, \varepsilon_t),$$

subject to appropriate technical conditions. Here $b, f : \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ are functions satisfying assumptions to be specified later. Consequently, the game (N-BD) can be analyzed by studying the following MFG counterpart.

MFG of bounded velocity. It is

$$\begin{aligned} v_{T,\theta}(s,x) &:= \inf_{(\xi_\cdot^+,\xi_\cdot^-)\in\mathcal{U}_{T,\theta}} J_{T,\theta}^\infty(s,x,\xi_\cdot^+,\xi_\cdot^-) \\ &= \inf_{(\xi_\cdot^+,\xi_\cdot^-)\in\mathcal{U}_{T,\theta}} \mathbb{E}\left[\int_s^T \left(f(x_t,\mu_t)+\gamma_1\dot{\xi}_t^+ + \gamma_2\dot{\xi}_t^-\right)dt\right], \end{aligned} \tag{MFG-BD}$$

subject to

$$dx_t = \left(b(x_t,\mu_t)+\dot{\xi}_t^+ - \dot{\xi}_t^-\right)dt + \sigma dW_t, \quad x_s = x, \mu_s = \mu. \tag{4}$$

Here μ_t is a probability measure of x_t for any $t \in [s,T]$, and

$$\mathcal{U}_{T,\theta} = \Bigg\{(\xi_\cdot^+,\xi_\cdot^-)\Bigg|\xi_t^+ \text{ and } \xi_t^- \text{ are } \mathcal{F}_t^{(x_t,\mu_t)}\text{-adapted, cádlág, nondecreasing, } \xi_s^+ = \xi_s^- = 0,$$
$$0 \le \dot{\xi}_t^+, \dot{\xi}_t^- \le \theta, \mathbb{E}\left[\int_s^T d\xi_t^+\right] < \infty, \text{ and } \mathbb{E}\left[\int_s^T d\xi_t^-\right] < \infty, \text{ for } 0 \le s \le t \le T\Bigg\},$$

with $\{\mathcal{F}_t^{(x_t,\mu_t)}\}_{s\le t\le T}$ the natural filtration of $\{x_t,\mu_t\}_{s\le t\le T}$. Note that the drift term in the dynamics and the objective function now rely only on the local information x_t^i and the aggregated mean information process μ_t.

2.3 Solution framework

There are several criteria to analyze stochastic games. Two standard ones are the Pareto Optimality and the Nash Equilibrium (NE). In this paper we will focus on the NE. Depending on the problem setting and in particular the admissible controls, there are several forms of NEs, including the open loop NE, the closed loop NE, and the closed loop in feedback form NE (a.k.a., the Markovian NE).

Throughout the paper, we will consider the Markovian NE. Markovian NE means that the controls are deterministic functions of time t, current state x_t, and a fixed control μ_t. More precisely,

Definition 1 (Control function)**.** *A control of bounded velocity ξ_t is called* Markovian *if $d\xi_t = \dot{\xi}_t dt = \varphi(t, x_t; \{\mu_t\})dt$ for some function $\varphi : [0,T] \times \mathbb{R} \to \mathbb{R}$. $\varphi(t,x;\{\mu_t\})$ is called the* control function *for the fixed $\{\mu_t\}$. A control of a finite variation ξ_t is called* Markovian *if $d\xi_t = d\varphi(t,x;\{\mu_t\})$ for some function φ. φ is called the* control function *for the fixed $\{\mu_t\}$.*

Definition 2 (Markovian ϵ-Nash equilibrium to (N-FV))**.** *A Markovian control $(\xi_\cdot^{i*+},\xi_\cdot^{i*-}) \in \mathcal{U}_{T,\infty}^N$ for $i = 1,\dots,N$ is a* Markovian ϵ-Nash equilibrium *to* (N-FV) *if for any $i \in \{1,\dots,N\}$, any $(s,x) \in [0,T]\times\mathbb{R}$ and any Markovian $(\xi_\cdot^{i'+},\xi_\cdot^{i'-}) \in \mathcal{U}_{T,\infty}^N$,*

$$J_{T,\infty}^{i,N}(s,x,\xi_\cdot^{i'+},\xi_\cdot^{i'-};\xi_\cdot^{*-i}) \ge J_{T,\infty}^{i,N}(s,x,\xi_\cdot^{i*+},\xi_\cdot^{i*-};\xi_\cdot^{*-i}) - \epsilon.$$

Definition 3 (Markovian ϵ-Nash equilibrium to (N-BD))**.** *A Markovian control $(\xi_\cdot^{i*+},\xi_\cdot^{i*-}) \in \mathcal{U}_{T,\theta}^N$ for $i = 1,\dots,N$ is a* Markovian ϵ-Nash equilibrium *to* (N-BD) *if for any $i \in \{1,\dots,N\}$, any $(s,x) \in [0,T]\times\mathbb{R}$ and any Markovian $(\xi_\cdot^{i'+},\xi_\cdot^{i'-}) \in \mathcal{U}_{T,\theta}^N$,*

$$J_{T,\theta}^{i,N}(s,x,\xi_\cdot^{i'+},\xi_\cdot^{i'-};\xi_\cdot^{*-i}) \ge J_{T,\theta}^{i,N}(s,x,\xi_\cdot^{i*+},\xi_\cdot^{i*-};\xi_\cdot^{*-i}) - \epsilon.$$

The solution to MFGs (MFG-BD) is defined as follows;

Definition 4. *A solution of* (MFG-BD) *is a pair of Markovian control* $(\xi_\cdot^{*+}, \xi_\cdot^{*-}) \in \mathcal{U}_{T,\theta}$ *and a flow of probability measures* $\{\mu_t^*\} \in \mathcal{M}_{[0,T]}$ *satisfying* $v_{T,\theta}(s,x) = J_{T,\theta}^{\infty}(s,x,\xi_\cdot^{*+}, \xi_\cdot^{*-})$ *for all* $(s,x) \in [0,T] \times \mathbb{R}$*, with* μ_t^* *a probability measure of the optimally controlled process* x_t^* *for each* $t \in [0,T]$*, where the dynamics of* $\{x_t^*\}$ *is*

$$dx_t^* = \left(b(x_t^*, \mu_t^*) + \dot{\xi}_t^{*+} - \dot{\xi}_t^{*-}\right) dt + \sigma dW_t, \quad s \le t \le T, \quad x_s^* = x.$$

The analysis of stochastic games (N-FV), (N-BD), and (MFG-BD) is built on the PDE/control methodology of [21] and [19]. In this approach, the MFG is essentially analyzed by studying two coupled PDEs, the backward HJB equation and the forward SDE.

To start, we introduce the underlying stochastic control problems.

2.4 Corresponding control problems

Control problem of a bounded velocity with $T < \infty$. Let $\{\mu_t\} \in \mathcal{M}_{[0,T]}$ be a fixed exogenous flow of probability measures. Then, (MFG-BD) becomes the following control problem,

$$\begin{aligned} v_{T,\theta}(s,x;\{\mu_t\}) &\triangleq \inf_{(\xi_\cdot^+, \xi_\cdot^-) \in \mathcal{U}_{T,\theta}} J_{T,\theta}^{\infty}(s,x,\xi_\cdot^+, \xi_\cdot^-; \{\mu_t\}) \\ &= \inf_{(\xi_\cdot^+, \xi_\cdot^-) \in \mathcal{U}_{T,\theta}} \mathbb{E}\left[\int_s^T \left(f(x_t,\mu_t) + \gamma_1 \dot{\xi}_t^+ + \gamma_2 \dot{\xi}_t^-\right) dt\right], \end{aligned} \qquad \text{(Control-BD)}$$

subject to Eqn. (4). This is a classical stochastic control problem, and the corresponding HJB equation with the terminal condition is given by

$$\begin{aligned} -\partial_t v_{T,\theta} &= \inf_{\dot{\xi}^+, \dot{\xi}^- \in [0,\theta]} \left\{ \left(b(x,\mu) + (\dot{\xi}^+ - \dot{\xi}^-)\right) \partial_x v_{T,\theta} + \left(f(x,\mu) + \gamma_1 \dot{\xi}^+ + \gamma_2 \dot{\xi}^-\right) \right\} + \frac{\sigma^2}{2} \partial_{xx} v_{T,\theta} \\ &= \min\left\{ (\partial_x v_{T,\theta} + \gamma_1)\theta, (-\partial_x v_{T,\theta} + \gamma_2)\theta, 0 \right\} + b(x,\mu)\partial_x v_{T,\theta} + f(x,\mu) + \frac{\sigma^2}{2}\partial_{xx} v_{T,\theta}. \\ &\text{with } v_{T,\theta}(T,x;\{\mu_t\}) = 0, \quad \forall x \in \mathbb{R}. \end{aligned} \qquad (5)$$

If the controls are of finite variation, then we have

Control problem of a finite variation with $T < \infty$.

$$v_{T,\infty}(s,x;\{\mu_t\}) \triangleq \inf_{(\xi_\cdot^+, \xi_\cdot^-) \in \mathcal{U}_{T,\infty}} \mathbb{E}\left[\int_s^T f(x_t,\mu_t)dt + \gamma_1 d\xi_t^+ + \gamma_2 d\xi_t^-\right], \qquad \text{(Control-FV)}$$

subject to

$$dx_t = b(x_t,\mu_t)dt + \sigma dW_t + d\xi_t^+ - d\xi_t^-, \quad x_{s-} = x.$$

2.5 Main results

The main results in the paper are derived under the following technical assumptions.

Assumptions.

(A1). $b(x,\mu)$ and $f(x,\mu)$ are Lipschitz continuous in x and μ, and $b(x,\mu)$ is bounded. That is, $|b(x_1,\mu^1)-b(x_2,\mu^2)|\le Lip(b)(|x_1-x_2|+D^1(\mu^1,\mu^2))$ for some $Lip(b)>0$, and $|f(x_1,\mu^1)-f(x_2,\mu^2)|\le Lip(f)(|x_1-x_2|+D^1(\mu^1,\mu^2))$ for some $Lip(f)>0$, and $|b(x,\mu)|\le c_1$ for some c_1.

(A2). $f(x,\mu)$ has a first-order derivative in x with $f(x,\mu)$ and $\partial_x f(x,\mu)$ satisfying the polynomial growth condition. Moreover, for any fixed $\mu\in\mathcal{P}_1(\mathbb{R})$, $f(x,\mu)$ is convex and nonlinear in x.

(A3). $b(x,\mu)$ has first- and second-order derivatives with respect to x with uniformly continuous and bounded derivatives in x.

(A4). $-\gamma_1<\gamma_2$. This ensures the finiteness of the value function. Indeed, take game (N-FV) with $-\gamma_1>\gamma_2$. Then, letting $d\xi_t^{i+}=d\xi_t^{i-}=M$ and $M\to\infty$, we will have $J_{T,\infty}^{i,N}\to-\infty$.

(A5). (Monotonicity of the cost function) f satisfies either

$$\text{(i).}\int_{\mathbb{R}}(f(x,\mu^1)-f(x,\mu^2))(\mu^1-\mu^2)(dx)\ge 0,\ \text{for any}\ \mu^1,\mu^2\in\mathcal{P}_1(\mathbb{R}),$$

and $H(x,p)=\inf_{\dot{\xi}^+,\dot{\xi}^-\in[0,\theta]}\{(\dot{\xi}^+-\dot{\xi}^-)p+\gamma_1\dot{\xi}^++\gamma_2\dot{\xi}^-\}$ satisfies the following condition for any $x,p,q\in\mathbb{R}$

$$\text{if}\ H(x,p+q)-H(x,p)-\partial_pH(x,p)q=0,\ \text{then}\ \partial_pH(x,p+q)=\partial_pH(x,p),\quad\text{or}$$

$$\text{(ii).}\int_{\mathbb{R}}(f(x,\mu^1)-f(x,\mu^2))(\mu^1-\mu^2)(dx)>0,\ \text{for any}\ \mu^1\neq\mu^2\in\mathcal{P}_1(\mathbb{R}).$$

As in [21, 5], Assumption (A5) is critical to ensure the solution of (MFG-BD), as will be clear from the proof of Proposition 6 for the uniqueness of the fixed point.

(A6). (Rationality of players) For any control function φ, any $t\in[0,T]$, any fixed $\{\mu_t\}$, and any $x,y\in\mathbb{R}$, $(x-y)\Big(\varphi(t,x;\{\mu_t\})-\varphi(t,y;\{\mu_t\})\Big)\le 0$.

Intuitively, it says that the better off the state of an individual player, the less likely the player exercises controls, in order to minimize her cost.

(A7). For any fixed $\mu\in\mathcal{P}_1(\mathbb{R})$, there exist $b_1,b_2\in\mathbb{R}$ such that $b(x,\mu)\le b_1+b_2x$ and $b_2<r$ for any $x\in\mathbb{R}$.

(A8). There exists some constant c_f satisfying $|f(x,\mu)|\le c_f\Big(1+|x|^2+\int_{\mathbb{R}}y^2\mu(dy)\Big)$ for any $x\in\mathbb{R},\mu\in\mathcal{P}_1(\mathbb{R})$.

Theorem 1. *Assume* (A1)–(A4). *Then for any* $(s,x)\in[0,T]\times\mathbb{R}$, *as* $\theta\to\infty$, *the value function* $v_{T,\theta}(s,x;\{\mu_t\})$ *of* (Control-BD) *converges to the value function* $v_{T,\infty}(s,x;\{\mu_t\})$ *of* (Control-FV).

Theorem 2. *Assume* (A1)–(A6). *Then there exists a unique solution* $((\xi_\cdot^{*+},\xi_\cdot^{*-}),\{\mu_t^*\})$ *of* (MFG-BD). *Moreover, the corresponding value function* $v_{T,\theta}(s,x)$ *for* (MFG-BD) *is in* $C^{1,2}([0,T]\times\mathbb{R})$ *with a polynomial growth.*

Theorem 3. *Assume* (A1)–(A6)*. Then, the optimal control to* (MFG-BD) *is an*

a). ϵ_N*-Nash equilibrium to* (N-BD)*, with* $\epsilon_N = O\left(\frac{1}{\sqrt{N}}\right)$*;*

b). $(\epsilon_N + \epsilon_\theta)$*-Nash equilibrium to* (N-FV)*, with* $\epsilon_N = O\left(\frac{1}{\sqrt{N}}\right)$ *and* $\epsilon_\theta \to 0$ *as* $\theta \to \infty$*.*

3 Proofs

The proof of the main results relies on the analysis of their perspective underlying control problems, in particular problem (Control-BD).

3.1 Analysis of problem (Control-BD)

To study problem (Control-BD), let us recall the viscosity solution to the HJB Eqn. (5).

Definition 5. $\hat{v}$ *is called a* viscosity solution *to the HJB Eqn.* (5) *if* $\hat{v}$ *is both a viscosity supersolution and a viscosity subsolution, with the following definitions,*
(i). viscosity supersolution*: for any* $(t_0, x_0) \in [0,T] \times \mathbb{R}$ *and any* $\vartheta \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R})$*, if* (t_0, x_0) *is a local minimum of* $\hat{v} - \vartheta$ *with* $\hat{v}(t_0, x_0) - \vartheta(t_0, x_0) = 0$*, then*

$$
\begin{aligned}
-\partial_t \vartheta(t_0, x_0) - \inf_{\dot{\xi}^+, \dot{\xi}^- \in [0,\theta]} &\left\{ \left(b(x_0, \mu) + (\dot{\xi}^+ - \dot{\xi}^-) \right) \partial_x \vartheta(t_0, x_0) + \left(f(x_0, \mu) + \gamma_1 \dot{\xi}^+ + \gamma_2 \dot{\xi}^- \right) \right\} \\
&- \frac{\sigma^2}{2} \partial_{xx} \vartheta(t_0, x_0) \geq 0, \qquad \text{and} \quad \vartheta(T, x_0) \geq 0;
\end{aligned}
$$

(ii). viscosity subsolution*: for any* $(t_0, x_0) \in [0,T] \times \mathbb{R}$ *and any* $\vartheta \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R})$*, if* (t_0, x_0) *is a local maximum of* $\hat{v} - \vartheta$ *with* $\hat{v}(t_0, x_0) - \vartheta(t_0, x_0) = 0$*, then*

$$
\begin{aligned}
-\partial_t \vartheta(t_0, x_0) - \inf_{\dot{\xi}^+, \dot{\xi}^- \in [0,\theta]} &\left\{ \left(b(x_0, \mu) + (\dot{\xi}^+ - \dot{\xi}^-) \right) \partial_x \vartheta(t_0, x_0) + \left(f(x_0, \mu) + \gamma_1 \dot{\xi}^+ + \gamma_2 \dot{\xi}^- \right) \right\} \\
&- \frac{\sigma^2}{2} \partial_{xx} \vartheta(t_0, x_0) \leq 0, \qquad \text{and} \quad \vartheta(T, x_0) \leq 0.
\end{aligned}
$$

Proposition 1. *Assume* (A1)–(A4)*. The HJB Eqn.* (5) *has a unique solution* v *in* $C^{1,2}([0,T] \times \mathbb{R})$ *with a polynomial growth. Furthermore, this solution is the value function to problem* (Control-BD)*, and the corresponding optimal control function is*

$$
\varphi_\theta(t, x_t; \{\mu_t\}) = \dot{\xi}^+_{t,\theta} - \dot{\xi}^-_{t,\theta} = \begin{cases} \theta & \text{if} \quad \partial_x v_{T,\theta}(t, x_t; \{\mu_t\}) \leq -\gamma_1, \\ 0 & \text{if} \quad -\gamma_1 < \partial_x v_{T,\theta}(t, x_t; \{\mu_t\}) < \gamma_2, \\ -\theta & \text{if} \quad \gamma_2 \leq \partial_x v_{T,\theta}(t, x_t; \{\mu_t\}). \end{cases} \tag{6}
$$

Proof. By [13, Theorem 6.2, Chapter VI], the HJB Eqn. (5) has a unique solution ϑ in $C^{1,2}([0,T] \times \mathbb{R})$ with a polynomial growth. Now we show that it is the value function to problem (Control-BD).

First, by the viscosity subsolution property of ϑ, for any $(s, x) \in [0,T] \times \mathbb{R}$, $\vartheta(T, x) \leq 0$ and

$$
-\partial_t \vartheta - \inf_{\dot{\xi}^+, \dot{\xi}^- \in [0,\theta]} \left\{ \left(b(x, \mu) + (\dot{\xi}^+ - \dot{\xi}^-) \right) \partial_x \vartheta + \left(f(x, \mu) + \gamma_1 \dot{\xi}^+ + \gamma_2 \dot{\xi}^- \right) \right\} - \frac{\sigma^2}{2} \partial_{xx} \vartheta \leq 0.
$$

Moreover, for any $(\dot{\xi}^+_\cdot, \dot{\xi}^-_\cdot) \in \mathcal{U}_{T,\theta}$, let x_t be a controlled process with $(\dot{\xi}^+_\cdot, \dot{\xi}^-_\cdot)$. Then, applying Itô's formula to $\vartheta(s,x)$,

$$\begin{aligned}
&0 \geq \mathbb{E}[\vartheta(T, x_T)] \\
&= \vartheta(s,x) + \mathbb{E}\left[\int_s^T \partial_t \vartheta(t,x_t) + (b(x_t,\mu_t) + (\dot{\xi}^+_t - \dot{\xi}^-_t))\partial_x \vartheta(t,x_t) + \frac{\sigma^2}{2}\partial_{xx}\vartheta(t,x_t)dt\right] \\
&\geq \vartheta(s,x) - \mathbb{E}\left[\int_s^T f(x_t,\mu_t) + \gamma_1 \dot{\xi}^+_t + \gamma_2 \dot{\xi}^-_t dt\right].
\end{aligned}$$

Hence, for any $(\xi^+_\cdot, \xi^-_\cdot) \in \mathcal{U}_{T,\theta}$, $\mathbb{E}\left[\int_s^T f(x_t,\mu_t) + \gamma_1 \dot{\xi}^+_t + \gamma_2 \dot{\xi}^-_t dt\right] \geq \vartheta(s,x)$. Therefore,

$$v_{T,\theta}(s,x;\{\mu_t\}) = \inf_{(\xi^+_\cdot,\xi^-_\cdot)\in\mathcal{U}_{T,\theta}} \mathbb{E}\left[\int_s^T f(x_t,\mu_t) + \gamma_1 \dot{\xi}^+_t + \gamma_2 \dot{\xi}^-_t dt\right] \geq \vartheta(s,x).$$

Furthermore, let $\{x_{t,\theta}\}$ be the controlled process under the control $(\xi^+_{\cdot,\theta}, \xi^-_{\cdot,\theta})$. By definition of $(\xi^+_{\cdot,\theta}, \xi^-_{\cdot,\theta})$,

$$-\partial_t\vartheta - \left\{\left(b(x_{t,\theta},\mu_t) + (\dot{\xi}^+_{t,\theta} - \dot{\xi}^-_{t,\theta})\right)\partial_x\vartheta + \left(f(x_{t,\theta},\mu_t) + \gamma_1\dot{\xi}^+_{t,\theta} + \gamma_2\dot{\xi}^-_{t,\theta}\right)\right\} - \frac{\sigma^2}{2}\partial_{xx}\vartheta = 0.$$

Then, applying Itô's formula to $\vartheta(t,x)$,

$$\begin{aligned}
&0 = \mathbb{E}[\vartheta(T, x_{T,\theta})] \\
&= \vartheta(s,x) + \mathbb{E}\left[\int_s^T \left(\partial_t \vartheta(t,x_{t,\theta}) + (b(x_{t,\theta},\mu_t) + (\dot{\xi}^+_{t,\theta} - \dot{\xi}^-_{t,\theta})\partial_x \vartheta(t,x_{t,\theta}) + \frac{\sigma^2}{2}\partial_{xx}\vartheta(t,x_{t,\theta})\right)dt\right] \\
&= \vartheta(s,x) - \mathbb{E}\left[\int_s^T \left(f(x_{t,\theta},\mu_t) + \gamma_1 \dot{\xi}^+_{t,\theta} + \gamma_2 \dot{\xi}^-_{t,\theta}\right) dt\right] \leq \vartheta(s,x) - v_{T,\theta}(s,x;\{\mu_t\}).
\end{aligned}$$

Hence, $v(s,x;\{\mu_t\}) \leq \vartheta(s,x)$.

Combined, $v_{T,\theta}(s,x;\{\mu_t\}) = \vartheta(s,x)$, and the optimal control function is

$$\varphi_\theta(t,x_t;\{\mu_t\}) = \begin{cases} \theta & \text{if} \quad \partial_x v_{T,\theta}(t,x_{t,\theta};\{\mu_t\}) \leq -\gamma_1, \\ 0 & \text{if} \quad -\gamma_1 < \partial_x v_{T,\theta}(t,x_{t,\theta};\{\mu_t\}) < \gamma_2, \\ -\theta & \text{if} \quad \gamma_2 \leq \partial_x v_{T,\theta}(t,x_{t,\theta};\{\mu_t\}). \end{cases} \qquad \square$$

Next, we establish the regularity of the value function to problem (Control-BD).

Proposition 2. *Assume* (A1)–(A4). *For any fixed $t \in [0,T]$, the value function $v_{T,\theta}(t,x;\{\mu_t\})$ is strictly convex in x.*

Proof. Fix any $x_1, x_2 \in \mathbb{R}$ and any $\lambda \in [0,1]$. For any $(\xi^{1,+}_\cdot, \xi^{1,-}_\cdot) \in \mathcal{U}_{T,\theta}$ and $(\xi^{2,+}_\cdot, \xi^{2,-}_\cdot) \in \mathcal{U}_{T,\theta}$, by the convexity of f,

$$\begin{aligned}
&\lambda J^\infty_{T,\theta}(s,x_1,\xi^{1,+}_\cdot,\xi^{1,-}_\cdot;\{\mu_t\}) + (1-\lambda)J^\infty_{T,\theta}(s,x_2,\xi^{2,+}_\cdot,\xi^{2,-}_\cdot;\{\mu_t\}) \\
\geq &J^\infty_{T,\theta}(s,\lambda x_1 + (1-\lambda)x_2, \lambda\xi^{1,+}_\cdot + (1-\lambda)\xi^{1,+}_\cdot, \lambda\xi^{2,+}_\cdot + (1-\lambda)\xi^{2,-}_\cdot;\{\mu_t\}) \\
\geq &v_{T,\theta}(s,\lambda x_1 + (1-\lambda)x_2;\{\mu_t\}).
\end{aligned}$$

Since this holds for any $(\xi^{1,+}_\cdot, \xi^{1,-}_\cdot) \in \mathcal{U}_{T,\theta}$ and $(\xi^{2,+}_\cdot, \xi^{2,-}_\cdot) \in \mathcal{U}_{T,\theta}$,

$$\lambda v_{T,\theta}(s,x_1;\{\mu_t\}) + (1-\lambda)J^\infty_{T,\theta}(s,x_2,\xi^{2,+}_\cdot,\xi^{2,-}_\cdot;\{\mu_t\}) \geq v_{T,\theta}(s,\lambda x_1 + (1-\lambda)x_2;\{\mu_t\}),$$

$$\lambda v_{T,\theta}(s,x_1;\{\mu_t\}) + (1-\lambda)v_{T,\theta}(s,x_2;\{\mu_t\}) \geq v_{T,\theta}(s,\lambda x_1 + (1-\lambda)x_2;\{\mu_t\}).$$

Hence, $v_{T,\theta}(s,x;\{\mu_t\})$ is convex in x. By Proposition 1, $v_{T,\theta}(s,x;\{\mu_t\})$ is a $\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ solution to the equation

$$-\partial_t v_{T,\theta} = \min\left\{(\partial_x v_{T,\theta}+\gamma_1)\theta, (-\partial_x v_{T,\theta}+\gamma_2)\theta, 0\right\} + b(x,\mu)\partial_x v_{T,\theta} + f(x,\mu) + \frac{\sigma^2}{2}\partial_{xx}v_\theta.$$

Since $f(x,\mu)$ is not linear in x, the solution to this equation is also nonlinear in x. Hence, $v_{T,\theta}(s,x;\{\mu_t\})$ is strictly convex. □

Proposition 3. *Assume* (A1)–(A4). *The optimal control function $\varphi_\theta(t,x;\{\mu_t\})$ to problem* (Control-BD) *is unique and the optimally controlled state process $x_{t,\theta}$ is also unique, with*

$$dx_{t,\theta} = \left(b(x_{t,\theta},\mu_t) + \varphi_\theta(t,x_{t,\theta};\{\mu_t\})\right)dt + \sigma dW_t, \quad x_{s,\theta} = x.$$

Proof. By Proposition 1, there exists a unique value function $v_{T,\theta}(t,x;\{\mu_t\})$ to problem (Control-BD). Furthermore, by (6), the optimal control function $\varphi_\theta(t,x;\{\mu_t\})$ is uniquely determined. Let us prove that the optimally controlled state process $x_{t,\theta}$ exists and is unique.

For any given $x^n_{t,\theta}$, consider a mapping Φ such that $\Phi(x^n_{t,\theta}) = x^{n+1}_{t,\theta}$ where $x^{n+1}_{t,\theta}$ is a solution to the following SDE:

$$dx^{n+1}_{t,\theta} = \left(b(x^n_{t,\theta},\mu_t) + \varphi_\theta(t,x^{n+1}_{t,\theta};\{\mu_t\})\right)dt + \sigma dW_t, \quad x^{n+1}_{s,\theta} = x. \tag{7}$$

By [29], for any given $x^n_{t,\theta}$, the SDE (7) has a unique solution $x^{n+1}_{t,\theta}$, so the mapping Φ is well defined. Then, for any $n \in \mathbb{N}$,

$$d(x^{n+1}_{t,\theta} - x^{n+2}_{t,\theta}) = \left(b(x^n_{t,\theta},\mu_t) - b(x^{n+1}_{t,\theta},\mu_t) + \varphi_\theta(t,x^{n+1}_{t,\theta};\{\mu_t\}) - \varphi_\theta(t,x^{n+2}_{t,\theta};\{\mu_t\})\right)dt.$$

Because $\varphi_\theta(t,x;\{\mu_t\})$ is nonincreasing in x,

$$\begin{aligned}
&d(x^{n+1}_{t,\theta} - x^{n+2}_{t,\theta})^2 \\
&= 2(x^{n+1}_{t,\theta} - x^{n+2}_{t,\theta})\left(b(x^n_{t,\theta},\mu_t) - b(x^{n+1}_{t,\theta},\mu_t) + \varphi_\theta(t,x^{n+1}_{t,\theta};\{\mu_t\}) - \varphi_\theta(t,x^{n+2}_{t,\theta};\{\mu_t\})\right)dt \\
&\leq 2Lip(b)|x^{n+1}_{t,\theta} - x^{n+2}_{t,\theta}||x^n_{t,\theta} - x^{n+1}_{t,\theta}|dt \\
&\leq Lip(b)\left(|x^{n+1}_{t,\theta} - x^{n+2}_{t,\theta}|^2 + |x^n_{t,\theta} - x^{n+1}_{t,\theta}|^2\right)dt.
\end{aligned}$$

By Gronwall's inequality, for any $t \in [0,T]$,

$$|x^{n+1}_{t,\theta} - x^{n+2}_{t,\theta}|^2 \leq Lip(b)\exp\left(Lip(b)t\right)\int_0^t |x^n_{s,\theta} - x^{n+1}_{s,\theta}|^2 ds.$$

Hence, we can derive for any $n \in \mathbb{N}$,

$$|x^{n+1}_{t,\theta} - x^{n+2}_{t,\theta}|^2 \leq \frac{\left(Lip(b)t\right)^n \exp\left(nLip(b)t\right)}{n!}|x^1_{t,\theta} - x^2_{t,\theta}|^2.$$

As $n \to \infty$, Φ is a contraction mapping, and the SDE (7) has a unique fixed point solution. Therefore, there exists a unique optimally controlled state process $x_{t,\theta}$ to problem (Control-BD). Furthermore, the optimal Markovian control $(\xi^+_{\cdot,\theta}, \xi^-_{\cdot,\theta})$ to (Control-BD) also uniquely exists. □

3.2 Proof of Theorem 1

Fix $\{\mu_t\} \in \mathcal{M}_{[0,T]}$. For any $(\zeta^+_{\cdot,\infty}, \zeta^-_{\cdot,\infty}) \in \mathcal{U}_{T,\infty}$, since each path of a finite variation process is almost everywhere differentiable, there exists a sequence of bounded velocity functions which converges to the path as $\theta \to \infty$. Hence, there exists a sequence $\{(\zeta^+_{\cdot,\theta}, \zeta^-_{\cdot,\theta})\}_{\theta\in[0,\infty)}$ such that $(\zeta^+_{\cdot,\theta}, \zeta^-_{\cdot,\theta}) \in \mathcal{U}_{T,\theta}$ and $\mathbb{E}\int_0^T |\dot{\zeta}^+_{t,\theta}dt - d\zeta^+_{t,\infty}| \to 0, \mathbb{E}\int_0^T |\dot{\zeta}^-_{t,\theta}dt - d\zeta^-_{t,\infty}| \to 0$ as $\theta \to \infty$. Define ϵ_θ as

$$\epsilon_\theta = O\bigg(\mathbb{E}\int_0^T |\dot{\zeta}^+_{t,\theta}dt - d\zeta^+_{t,\infty}| + \mathbb{E}\int_0^T |\dot{\zeta}^-_{t,\theta}dt - d\zeta^-_{t,\infty}|\bigg), \tag{8}$$

and $\epsilon_\theta \to 0$ as $\theta \to \infty$.

Denote

$$\begin{aligned} d\hat{x}_{t,\theta} &= (b(\hat{x}_{t,\theta}, \mu_t) + \dot{\zeta}^+_{t,\theta} - \dot{\zeta}^-_{t,\theta})dt + \sigma dW_t, \quad \hat{x}_{s,\theta} = x, \text{ and} \\ d\hat{x}_t &= b(\hat{x}_t, \mu_t)dt + \sigma dW_t + d\zeta^+_{t,\infty} - d\zeta^-_{t,\infty}, \qquad \hat{x}_{s-} = x. \end{aligned}$$

Then, for any $\tau \in [s, T]$,

$$\begin{aligned} |\hat{x}_{\tau,\theta} - \hat{x}_\tau| &\le \int_s^\tau |b(\hat{x}_{t,\theta}, \mu_t) - b(\hat{x}_t, \mu_t)|dt + \int_s^\tau |\dot{\zeta}^+_{t,\theta}dt - d\zeta^+_{t,\infty}| + \int_s^\tau |\dot{\zeta}^-_{t,\theta}dt - d\zeta^-_{t,\infty}| \\ &\le \int_s^\tau Lip(b)|\hat{x}_{t,\theta} - \hat{x}_t|dt + \int_s^\tau |\dot{\zeta}^+_{t,\theta}dt - d\zeta^+_{t,\infty}| + \int_s^\tau |\dot{\zeta}^-_{t,\theta}dt - d\zeta^-_{t,\infty}|. \end{aligned}$$

By Gronwall's inequality,

$$\mathbb{E}|\hat{x}_{\tau,\theta} - \hat{x}_\tau| \le O\left(\mathbb{E}\int_0^\tau |\dot{\zeta}^+_{t,\theta}dt - d\zeta^+_{t,\infty}| + \mathbb{E}\int_0^\tau |\dot{\zeta}^-_{t,\theta}dt - d\zeta^-_{t,\infty}|\right).$$

Consequently,

$$\begin{aligned} &\bigg|J^\infty_{T,\infty}(s, x, \zeta^+_{t,\infty}, \zeta^-_{t,\infty}; \{\mu_t\}) - J^\infty_{T,\theta}(s, x, \zeta^+_{t,\theta}, \zeta^-_{t,\theta}; \{\mu_t\})\bigg| \\ &\le \mathbb{E}\bigg[\bigg|\int_s^T f(\hat{x}_t, \mu_t) - f(\hat{x}_{t,\theta}, \mu_t) + \gamma_1 d\zeta^+_{t,\infty} + \gamma_2 d\zeta^-_{t,\infty} - \gamma_1\dot{\zeta}^+_{t,\theta}dt - \gamma_2\dot{\zeta}^-_{t,\theta}dt\bigg|\bigg] \\ &\le \mathbb{E}\bigg[\int_s^T Lip(f)|\hat{x}_t - \hat{x}_{t,\theta}| + \gamma_1|d\zeta^+_{t,\infty} - \dot{\zeta}^+_{t,\theta}dt| + \gamma_2|d\zeta^-_{t,\infty} - \dot{\zeta}^-_{t,\theta}dt|\bigg] \\ &\le O\bigg(\mathbb{E}\int_0^T |\dot{\zeta}^+_{t,\theta}dt - d\zeta^+_{t,\infty}| + \mathbb{E}\int_0^T |\dot{\zeta}^-_{t,\theta}dt - d\zeta^-_{t,\infty}|\bigg). \end{aligned}$$

Therefore, $\bigg|v_{T,\infty}(s, x; \{\mu_t\}) - v_{T,\theta}(s, x; \{\mu_t\})\bigg| \to 0$ as $\theta \to 0$.

3.3 Proof of Theorem 2

First, from Proposition 1 we see that for any given fixed $\{\mu_t\}$ there exists a unique optimal control function as $\varphi_\theta(t, x; \{\mu_t\})$. Now, one can define a mapping Γ_1 from $\mathcal{M}_{[0,T]}$ to a class of pairs of the optimal control function φ_θ and the fixed flow of probability measures $\{\mu_t\}$ such that

$$\Gamma_1(\{\mu_t\}) = \bigg(\varphi_\theta(t, x; \{\mu_t\}), \{\mu_t\}\bigg).$$

Moreover, by Proposition 3 the optimally controlled process $x_{t,\theta}$ under the fixed $\{\mu_t\}$ exists uniquely with

$$dx_{t,\theta} = \Big(b(x_{t,\theta}, \mu_t) + \varphi_\theta(t, x_{t,\theta}; \{\mu_t\})\Big)dt + \sigma dW_t, \qquad x_{s,\theta} = x.$$

Consequently, we can define Γ_2 so that

$$\Gamma_2\Big(\varphi_\theta(t, x; \{\mu_t\}), \{\mu_t\}\Big) = \{\tilde{\mu}_t\},$$

where $\tilde{\mu}_t$ is the probability measure of $x_{t,\theta}$ for each $t \in [0, T]$.

Now, define a mapping Γ as

$$\Gamma(\{\mu_t\}) = \Gamma_2 \circ \Gamma_1(\{\mu_t\}) = \{\tilde{\mu}_t\}.$$

We will use the Schauder fixed point theorem [26, Theorem 4.1.1] to show the existence of a fixed point. The key is to prove that Γ is a continuous mapping of $\mathcal{M}_{[0,T]}$ into $\mathcal{M}_{[0,T]}$, and the range of Γ is relatively compact.

Proposition 4. *Assume* (A1)–(A4). *Γ is a mapping from $\mathcal{M}_{[0,T]}$ to $\mathcal{M}_{[0,T]}$.*

Proof. For any $\{\mu_t\}$ in $\mathcal{M}_{[0,T]}$, let us prove that $\{\tilde{\mu}_t\} = \Gamma(\{\mu_t\})$ is also in $\mathcal{M}_{[0,T]}$. Without loss of generality, suppose $s > t$, and

$$x_s = x_t + \int_t^s \Big(b(x_r, \mu_r) + \varphi_\theta(r, x_r; \{\mu_t\})\Big)dr + \int_t^s \sigma dW_r.$$

Since $b(x, \mu)$ is Lipschitz, $|\varphi_\theta(s, x_s; \{\mu_t\})| \le \theta$, and $\mathbb{E}\Big|(b(x_r, \mu_r) + \varphi_\theta(r, x_r; \{\mu_t\}))\Big| \le M$ for large M and for any $r \in [0, T]$,

$$\begin{aligned} D^1(\tilde{\mu}_s, \tilde{\mu}_t) &\le \mathbb{E}|x_s - x_t| \\ &\le \mathbb{E}\int_t^s \Big|b(x_r, \mu_r) + \varphi(r, x_r; \{\mu_t\})\Big|dr + \sigma\mathbb{E}\sup_{r\in[t,s]}|W_r - W_t| \\ &\le M|s-t| + \sigma\mathbb{E}\sup_{r\in[t,s]}|W_r - W_t| \le M|s-t| + \sigma|s-t|^{\frac{1}{2}}. \end{aligned}$$

Therefore, $\sup_{s\neq t}\frac{D^1(\tilde{\mu}_t, \tilde{\mu}_s)}{|t-s|^{\frac{1}{2}}} \le c$. For any $t \in [0, T]$, since $|b(x, \mu)| \le c_1$ is bounded,

$$\int_{\mathbb{R}} |x|^2 \tilde{\mu}_t(dx) \le 2\mathbb{E}\Big[\int_{\mathbb{R}} |x|^2 d\tilde{\mu}_0 + c_2^2 t^2 + \sigma^2 t\Big] \le 2\mathbb{E}\Big[\int_{\mathbb{R}} |x|^2 d\tilde{\mu}_0 + c_1^2 T^2 + \sigma^2 T\Big],$$

and $\sup_{t\in[0,T]} \int_{\mathbb{R}} |x|^2 \tilde{\mu}_t(dx) \le c$. □

Proposition 5. *Assume* (A1)–(A6). *$\Gamma : \mathcal{M}_{[0,T]} \to \mathcal{M}_{[0,T]}$ is continuous.*

Proof. Let $\{\mu_t^n\} \in \mathcal{M}_{[0,T]}$ for $n = 1, \ldots,$ be a sequence of flows of probability measures $d_{\mathcal{M}}(\{\mu_t^n\}, \{\mu_t\}) \to 0$ as $n \to \infty$, for some $\{\mu_t\} \in \mathcal{M}_{[0,T]}$. Fix $\tau \in [0, T)$. By Proposition 1, for each $\{\mu_t^n\}$, problem

(Control-BD) has a value function $v^n_{T,\theta}(s,x;\{\mu^n_t\})$ with the optimal control $\varphi^n(t,x)$. [1] Let $\{x^n_t\}$ be the corresponding optimal controlled process:

$$dx^n_t = \Big(b(x^n_t,\mu^n_t) + \varphi^n(t,x^n_t)\Big)dt + \sigma dW_t, \quad \tau \le t \le T, \quad x^n_\tau = x.$$

Let $\{\tilde{\mu}^n_t\}$ be a flow of probability measures of $\{x^n_t\}$, then $\Gamma(\{\mu^n_t\}) = \{\tilde{\mu}^n_t\}$.

Similarly, for each $\{\mu_t\}$, problem (Control-BD) has a value function $v_{T,\theta}(s,x;\{\mu_t\})$ with the optimal control $\varphi(t,x)$. [2] Let $\{x_t\}$ be the corresponding optimal controlled process:

$$dx_t = \Big(b(x_t,\mu_t) + \varphi(t,x_t)\Big)dt + \sigma dW_t, \quad \tau \le t \le T, \quad x_\tau = x.$$

Let $\{\tilde{\mu}_t\}$ be a flow of probability measures of $\{x_t\}$, then $\Gamma(\{\mu_t\}) = \{\tilde{\mu}_t\}$.

To show that Γ is continuous, we need to show

$$d_{\mathcal{M}}\Big(\{\tilde{\mu}^n_t\},\{\tilde{\mu}_t\}\Big) \to 0 \text{ as } n \to \infty.$$

This is established in four steps.

Step 1. We first establish some relation between $D^2(\{\tilde{\mu}^n_t\},\{\tilde{\mu}_t\})$ and $D^2(\{\mu^n_t\},\{\mu_t\})$. For any $s \in [\tau, T]$,

$$d(x_s - x^n_s) = \Big(b(x_s,\mu_s) - b(x^n_s,\mu^n_s) + \varphi(s,x_s) - \varphi^n(s,x^n_s)\Big)ds.$$

Then, for any $t \in [\tau, T]$,

$$\begin{aligned}
|x_t - x^n_t|^2 &= 2\int_\tau^t \Big(b(x_s,\mu_s) - b(x^n_s,\mu^n_s) + \varphi(s,x_s) - \varphi^n(s,x^n_s)\Big)(x_s - x^n_s)ds \\
&\le 2\int_\tau^t Lip(b)\Big(|x_s - x^n_s| + D^1(\mu_s,\mu^n_s)\Big)|x_s - x^n_s| \\
&\quad + \Big(\varphi(s,x_s) - \varphi^n(s,x^n_s)\Big)(x_s - x^n_s)ds.
\end{aligned}$$

$$Lip(b)\Big(|x_s - x^n_s| + D^1(\mu_s,\mu^n_s)\Big)|x_s - x^n_s| \le Lip(b_0)|x_s - x^n_s|^2 + \frac{Lip(b)}{2}\Big((D^1(\mu_s,\mu^n_s))^2 + |x_s - x^n_s|^2\Big).$$

By Assumption (A6),

$$\begin{aligned}
&(\varphi(s,x^n_s) - \varphi^n(s,x^n_s))(x_s - x^n_s) \\
\le\ &\Big(\varphi(s,x_s) - \varphi^n(s,x_s) + \varphi^n(s,x_s) - \varphi^n(s,x^n_s)\Big)(x_s - x^n_s) \\
\le\ &(\varphi(s,x_s) - \varphi^n(s,x_s))(x_s - x^n_s) \\
\le\ &\frac{1}{2}\Big(|\varphi(s,x_s) - \varphi^n(s,x_s)|^2 + |x_s - x^n_s|^2\Big).
\end{aligned}$$

[1] In this proof, for simplicity of notation, denote $\varphi^n(t,x) = \varphi^n_\theta(t,x;\{\mu^n_t\})$.

[2] In this proof, for simplicity of notation, denote $\varphi(t,x) = \varphi_\theta(t,x;\{\mu_t\})$.

Consequently,

$$|x_t - x_t^n|^2 \leq \int_\tau^t (3Lip(b)+1)|x_s - x_s^n|^2 + Lip(b_0)(D^1(\mu_s,\mu_s^n))^2 + |\varphi(s,x_s) - \varphi^n(s,x_s)|^2 ds.$$

By Gronwall's inequality,

$$(D^2(\tilde{\mu}_t,\tilde{\mu}_t^n))^2 \leq c_2 \int_\tau^t Lip(b)(D^1(\mu_s,\mu_s^n))^2 + \mathbb{E}\Big[|\varphi(s,x_s) - \varphi^n(s,x_s)|^2\Big] ds, \tag{9}$$

for some constant c_2 depending on T and $Lip(b)$.

Step 2. Now we prove that for any $(t,x) \in [\tau,T]\times\mathbb{R}$,

$$\partial_x v_{T,\theta}^n(t,x;\{\mu_t^n\}) \to \partial_x v(t,x;\{\mu_t\}) \text{ as } n\to\infty.$$

By Proposition 1, $v_{T,\theta}$ and $v_{T,\theta}^n$ are the solutions to the HJB Eqn. (5). Denote

$$\varphi_1(s,x) = \max\{\varphi(s,x),0\}, \quad \varphi_2(s,x) = -\max\{-\varphi(s,x),0\},$$
$$\varphi_1^n(s,x) = \max\{\varphi^n(s,x),0\}, \quad \varphi_2^n(s,x) = -\max\{-\varphi^n(s,x),0\}.$$

Since φ_1,φ_2 are optimal controls, using Itô's formula and the HJB Eqn. (5), we obtain

$$\begin{aligned} &- v_{T,\theta}(\tau,x;\{\mu_t\}) \\ &= v_{T,\theta}(T,x_T;\{\mu_t\}) - v_{T,\theta}(\tau,x;\{\mu_t\}) \\ &= -\mathbb{E}\int_\tau^T \Big(f(x_s,\mu_s) + \gamma_1\varphi_1(s,x_s) + \gamma_2\varphi_2(s,x_s)\Big)ds + \int_\tau^T \sigma\partial_x v_{T,\theta}(s,x_s;\{\mu_t\})dW_s. \end{aligned} \tag{10}$$

Similarly, for any $n\in\mathbb{N}$, applying Itô's formula to $v_{T,\theta}^n(s,x)$ and $\{x_t\}$ yields

$$\begin{aligned} &v_{T,\theta}^n(T,x_T;\{\mu_t^n\}) - v_{T,\theta}^n(\tau,x;\{\mu_t^n\}) \\ =&\int_\tau^T \partial_t v_{T,\theta}^n(s,x_s;\{\mu_t^n\}) + (b(x_s,\mu_s)+\varphi(s,x_s))\partial_x v_{T,\theta}^n(s,x_s;\{\mu_t^n\}) + \frac{\sigma^2}{2}\partial_{xx} v_{T,\theta}^n(s,x_s;\{\mu_t^n\})ds \\ &+ \int_\tau^T \sigma\partial_x v_{T,\theta}^n(s,x_s;\{\mu_t^n\})dW_s \\ =&\int_\tau^T \partial_t v_{T,\theta}^n(s,x_s;\{\mu_t^n\}) + (b(x_s,\mu_s^n)+\varphi_{T,\theta}^n(s,x_s))\partial_x v_{T,\theta}^n(s,x_s;\{\mu_t^n\}) + \frac{\sigma^2}{2}\partial_{xx} v_{T,\theta}^n(s,x_s;\{\mu_t^n\})ds \\ &+ \int_\tau^T \sigma\partial_x v_{T,\theta}^n(s,x_s;\{\mu_t^n\})dW_s \\ &- \int_\tau^T (b(x_s,\mu_s^n) - b(x_s,\mu_s) + \varphi^n(s,x_s) - \varphi(s,x_s))\partial_x v_{T,\theta}^n(s,x_s;\{\mu_t^n\})ds \\ =&-\int_\tau^T \Big(f(x_s,\mu_s^n) + \gamma_1\varphi_1^n(s,x_s) + \gamma_2\varphi_2^n(s,x_s)\Big)ds + \int_\tau^T \sigma\partial_x v_{T,\theta}^n(s,x_s;\{\mu_t^n\})dW_s \\ &- \int_\tau^T (b(x_s,\mu_s^n) - b(x_s,\mu_s) + \varphi^n(s,x_s) - \varphi(s,x_s))\partial_x v_{T,\theta}^n(s,x_s;\{\mu_t^n\})ds. \end{aligned}$$

The last equality is due to the HJB Eqn. (5). Hence,

$$\begin{aligned} v_{T,\theta}^n(\tau,x;\{\mu_t^n\}) =& \int_\tau^T \Big(f(x_s,\mu_s^n) + \gamma_1\varphi_1^n(s,x_s) + \gamma_2\varphi_2^n(s,x_s)\Big)ds - \int_\tau^T \sigma\partial_x v_{T,\theta}^n(s,x_s;\{\mu_t^n\})dW_s \\ &+ \int_\tau^T \Big(b(x_s,\mu_s^n) - b(x_s,\mu_s) + \varphi^n(s,x_s) - \varphi(s,x_s))\partial_x v_{T,\theta}^n(s,x_s;\{\mu_t^n\}\Big)ds. \end{aligned} \tag{11}$$

Denote $H(s,x) = \inf_{\dot{\xi}^+,\dot{\xi}^- \in [0,\theta]} \{(\dot{\xi}^+ - \dot{\xi}^-)\partial_x v_{T,\theta}(s,x;\{\mu_t^n\}) + \gamma_1 \dot{\xi}^+ + \gamma_2 \dot{\xi}^-\}$, and
$H^n(s,x) = \inf_{\dot{\xi}^+,\dot{\xi}^- \in [0,\theta]} \{(\dot{\xi}^+ - \dot{\xi}^-)\partial_x v^n_{T,\theta}(s,x;\{\mu_t^n\}) + \gamma_1 \dot{\xi}^+ + \gamma_2 \dot{\xi}^-\}$. Then for any $\dot{\xi}^+, \dot{\xi}^- \in [0,\theta]$,

$$
\begin{aligned}
&\left|\Big((\dot{\xi}^+ - \dot{\xi}^-)\partial_x v_{T,\theta}(s,x;\{\mu_t\}) + \gamma_1 \dot{\xi}^+ + \gamma_2 \dot{\xi}^-\Big) - \Big((\dot{\xi}^+ - \dot{\xi}^-)\partial_x v^n_{T,\theta}(s,x;\{\mu_t^n\}) + \gamma_1 \dot{\xi}^+ + \gamma_2 \dot{\xi}^-\Big)\right| \\
&\leq \left|\dot{\xi}^+\Big(\partial_x v_{T,\theta}(s,x;\{\mu_t\}) - \partial_x v^n_{T,\theta}(x,\mu^n;\{\mu_t^n\})\Big) - \dot{\xi}^-\Big(\partial_x v_{T,\theta}(s,x;\{\mu_t\}) - \partial_x v^n_{T,\theta}(x,\mu^n;\{\mu_t^n\})\Big)\right| \\
&\leq 2\theta \left|\partial_x v_{T,\theta}(s,x;\{\mu_t\}) - \partial_x v^n_{T,\theta}(s,x;\{\mu_t^n\})\right|.
\end{aligned}
$$

Hence, for any $s,x \in [\tau,T] \times \mathbb{R}$,

$$
|H(s,x) - H^n(s,x)| \leq 2\theta \Big|\partial_x v_{T,\theta}(s,x;\{\mu_t\}) - \partial_x v^n_{T,\theta}(s,x;\{\mu_t^n\})\Big|.
$$

By definition,

$$
\begin{aligned}
&2\theta \Big|\partial_x v_{T,\theta}(s,x;\{\mu_t\}) - \partial_x v^n_{T,\theta}(s,x;\{\mu_t^n\})\Big| \\
&\geq \Big|\Big(\varphi_1(t,x) - \varphi_2(t,x)\Big)\partial_x v_{T,\theta}(s,x;\{\mu_t\}) + \gamma_1\varphi_1(t,x) + \gamma_2\varphi_2(t,x) \\
&\quad - \Big(\varphi_1^n(t,x) - \varphi_2^n(t,x)\Big)\partial_x v^n_{T,\theta}(s,x;\{\mu_t^n\}) + \gamma_1\varphi_1^n(t,x) + \gamma_2\varphi_2^n(t,x)\Big| \\
&= \Big|\Big(\gamma_1 + \partial_x v_{T,\theta}(s,x;\{\mu_t\})\Big)\Big(\varphi_1(t,x) - \varphi_1^n(t,x)\Big) + \Big(\gamma_2 - \partial_x v_{T,\theta}(s,x;\{\mu_t^n\})\Big)\Big(\varphi_2(t,x) - \varphi_2^n(t,x)\Big) \\
&\quad + \Big(\partial_x v_{T,\theta}(s,x;\{\mu_t\}) - \partial_x v^n_{T,\theta}(s,x;\{\mu_t^n\})\Big)\Big(\varphi_1^n(t,x) - \varphi_2^n(t,x)\Big)\Big| \\
&\geq \Big|\Big(\gamma_1 + \partial_x v_{T,\theta}(s,x;\{\mu_t\})\Big)\Big(\varphi_1(t,x) - \varphi_1^n(t,x)\Big) + \Big(\gamma_2 - \partial_x v_{T,\theta}(s,x;\{\mu_t\})\Big)\Big(\varphi_2(t,x) - \varphi_2^n(t,x)\Big)\Big| \\
&\quad - \theta\Big|\partial_x v_{T,\theta}(s,x;\{\mu_t\}) - \partial_x v^n_{T,\theta}(s,x;\{\mu_t^n\})\Big|.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
3\theta\Big|\partial_x v_{T,\theta}(s,x;\{\mu_t\}) - \partial_x v^n_{T,\theta}(s,x;;\{\mu_t^n\})\Big| \geq \Big|&\Big(\gamma_1 + \partial_x v_{T,\theta}(s,x;\{\mu_t\})\Big)\Big(\varphi_1(s,x) - \varphi_1^n(s,x)\Big) \\
&+ \Big(\gamma_2 - \partial_x v_{T,\theta}(s,x;\{\mu_t\})\Big)\Big(\varphi_2(s,x) - \varphi_2^n(s,x)\Big)\Big|.
\end{aligned} \tag{12}
$$

Similarly,

$$
\begin{aligned}
3\theta\Big|\partial_x v_{T,\theta}(s,x;\{\mu_t\}) - \partial_x v^n_{T,\theta}(s,x;\{\mu_t^n\})\Big| \geq \Big|&\Big(\gamma_1 + \partial_x v^n_{T,\theta}(s,x;\{\mu_t^n\})\Big)\Big(\varphi_1(s,x) - \varphi_1^n(s,x)\Big) \\
&+ \Big(\gamma_2 - \partial_x v^n_{T,\theta}(s,x;\{\mu_t^n\})\Big)\Big(\varphi_2(s,x) - \varphi_2^n(s,x)\Big)\Big|.
\end{aligned} \tag{13}
$$

Step 3. We can further show $\varphi^n(s,x) \to \varphi(s,x)$ for any $s,x \in [0,T] \times \mathbb{R}$ as $n \to \infty$.

Indeed, from Eqns. (10) and (11) and by Itô's isometry and Cauchy–Schwartz inequality,

$$
\begin{aligned}
&\Big(v_{T,\theta}(\tau,x;\{\mu_t\})-v^n_{T,\theta}(\tau,x;\{\mu^n_t\})\Big)^2+\sigma^2\mathbb{E}\Big[\int_\tau^T\Big(\partial_x v_{T,\theta}(s,x_s;\{\mu_t\})-\partial_x v^n_{T,\theta}(s,x_s;\{\mu^n_t\})\Big)^2 ds\Big]\\
\le&3(T-\tau)\mathbb{E}\Big[\int_\tau^T\Big(f(x_s,\mu_s)-f(x_s,\mu^n_s)\Big)^2+\Big((b(x_s,\mu_s)-b(x_s,\mu^n_s))\partial_x v^n_{T,\theta}(s,x_s;\{\mu_t\})\Big)^2\\
&+\Big((\gamma_1+\partial_x v^n_{T,\theta}(s,x;\{\mu^n_t\}))(\varphi_1(s,x_s)-\varphi^n_1(s,x_s))\\
&+(\gamma_2-\partial_x v^n_{T,\theta}(s,x;\{\mu^n_t\}))(\varphi_2(s,x_s)-\varphi^n_2(s,x_s))\Big)^2 ds\Big]\\
\le&3(T-\tau)\mathbb{E}\Big[\int_\tau^T\Big(Lip(f)D^1(\mu_s,\mu^n_s)\Big)^2+\Big(Lip(b)D^1(\mu_s,\mu^n_s)\Big|\partial_x v^n_{T,\theta}(s,x_s;\{\mu^n_t\})\Big|\Big)^2\\
&+\Big((\gamma_1+\partial_x v^n_{T,\theta}(s,x_s;\{\mu^n_t\}))(\varphi_1(s,x_s)-\varphi^n_1(s,x_s))\\
&+(\gamma_2-\partial_x v^n_{T,\theta}(s,x_s;\{\mu^n_t\}))(\varphi_2(s,x_s)-\varphi^n_2(s,x_s))\Big)^2 ds\Big]\\
\le&3(T-\tau)\mathbb{E}\Big[\int_\tau^T\Big(Lip(f)D^1(\mu_s,\mu^n_s)\Big)^2+\Big(Lip(b)D^1(\mu_s,\mu^n_s)|\partial_x v^n_{T,\theta}(s,x_s;\{\mu^n_t\})|\Big)^2\\
&+\Big(3\theta(\partial_x v_{T,\theta}(s,x_s;\{\mu^n_t\})-\partial_x v^n_{T,\theta}(s,x_s;\{\mu^n_t\}))\Big)^2 ds\Big].
\end{aligned}
$$

Let $\delta=\frac{\sigma^2}{54\theta^2}$. Then, for any $\tau\in[T-\delta,T]$,

$$
\begin{aligned}
&\Big(v_{T,\theta}(\tau,x;\{\mu_t\})-v^n_{T,\theta}(\tau,x;\{\mu^n_t\})\Big)^2+\frac{\sigma^2}{2}\mathbb{E}\Big[\int_\tau^T(\partial_x v_{T,\theta}(s,x_s;\{\mu_t\})-\partial_x v^n_{T,\theta}(s,x_s;\{\mu^n_t\}))^2 ds\Big]\\
\le&3(T-\tau)\mathbb{E}\Big[\int_\tau^T\Big(Lip(f)D^1(\mu_s,\mu^n_s)\Big)^2+\Big(Lip(b)D^1(\mu_s,\mu^n_s)|\partial_x v^n(s,x_s;\{\mu^n_t\})|\Big)^2 ds\Big].
\end{aligned}
$$

Hence, for any $\tau\in[T-\delta,T]$,

$$v_{T,\theta}(\tau,x;\{\mu_t\})-v^n_{T,\theta}(\tau,x;\{\mu^n_t\})\to 0,$$

and

$$\mathbb{E}\Big[\int_\tau^T\Big(\partial_x v_{T,\theta}(s,x_s;\{\mu_t\})-\partial_x v^n_{T,\theta}(s,x_s;\{\mu^n_t\})\Big)^2 ds\Big]\to 0 \text{ as } n\to\infty.$$

Since $\delta>0$, one can repeat this process for $[T-2\delta,T-\delta]$. Proceeding recursively, one can show that for any $(t,x)\in[0,T]\times\mathbb{R}$, $v^n_{T,\theta}(t,x;\{\mu^n_t\})\to v_{T,\theta}(t,x;\{\mu_t\})$, and $\mathbb{E}\Big[\int_0^T\Big(\partial_x v_{T,\theta}(s,x_s;\{\mu_t\})-\partial_x v^n_{T,\theta}(s,x_s;\{\mu^n_t\})\Big)^2 ds\Big]\to 0$ as $n\to\infty$. Hence, for any $(s,x)\in[0,T]\times\mathbb{R}$,

$$\partial_x v^n_{T,\theta}(s,x;\{\mu^n_t\})\to\partial_x v_{T,\theta}(s,x;\{\mu_t\}) \text{ as } n\to\infty.$$

Since $v^n_{T,\theta}(s,x;\{\mu^n_t\})$ and $v_{T,\theta}(s,x;\{\mu_t\})$ are convex in x and $f(x,\mu)$ is nonlinear, $v^n_{T,\theta}(s,x;\{\mu^n_t\})$ and $v_{T,\theta}(s,x;\{\mu_t\})$ are strictly convex. So, $\partial_x v^n_{T,\theta}(s,x;\{\mu^n_t\}),\partial_x v_{T,\theta}(s,x;\{\mu_t\})$ are strictly increasing in x, and by definition of φ^n and φ, $\varphi^n(s,x)$ converges to $\varphi(s,x)$ for any $(s,x)\in[0,T]\times\mathbb{R}$.

Step 4. We are now ready to show $d_{\mathcal{M}}\left(\{\tilde{\mu}_t\}, \{\tilde{\mu}_t^n\}\right) \to 0$ as $n \to \infty$.

From previous steps, $\varphi^n(s, x_s) \to \varphi(s, x_s)$ a.s. as $n \to \infty$, and by the Dominated Convergence Theorem in the L^2 space, for each $s \in [0, T]$, $\mathbb{E}\left|\varphi^n(s, x_s) - \varphi^n(s, x_s)\right|^2 \to 0$. Hence, by inequality (9), $D^2(\tilde{\mu}_t, \tilde{\mu}_t^n) \to 0$ for any $t \in [0, T]$. Since $D^1(\tilde{\mu}_t, \tilde{\mu}_t^n) \leq D^2(\tilde{\mu}_t, \tilde{\mu}_t^n)$, $D^1(\tilde{\mu}_t, \tilde{\mu}_t^n) \to 0$ for any $t \in [0, T]$, $d_{\mathcal{M}}\left(\{\tilde{\mu}_t\}, \{\tilde{\mu}_t^n\}\right) \to 0$ as $n \to \infty$. That is, Γ is continuous. □

Proposition 6. *Assume* (A1)–(A6). $\Gamma : \mathcal{M}_{[0,T]} \to \mathcal{M}_{[0,T]}$ *has a fixed point, and* (MFG-BD) *has a unique solution.*

Proof. As in the proof in Section 3.2 and the proof of Lemma 5.7 in [5], the range of the mapping Γ is relatively compact, and by Proposition 5, Γ is a continuous mapping. Hence, due to the Schauder fixed point theorem [26, Theorem 4.1.1], Γ has a fixed point such that $\Gamma(\{\mu_t\}) = \{\mu_t\} \in \mathcal{M}_{[0,T]}$. By Assumption (A5), the fixed point is at most one ([5, 21]). Therefore, there exists a unique fixed point solution of flow of probability measures $\{\mu_t^*\}$. By definition of the solution to a MFG and Proposition 1, the optimal control is also unique. □

3.4 Proof of Theorem 3

Suppose that $\left((\xi^+_{\cdot,\theta}, \xi^-_{\cdot,\theta}), \{\mu_{t,\theta}\}\right)$ is an MFG solution to (MFG-BD) with a bound θ, and $x_{t,\theta}$ is the optimally controlled process:

$$dx_{t,\theta} = \Big(b(x_{t,\theta}, \mu_{t,\theta}) + \varphi_{1,\theta}(t, x_{t,\theta}; \{\mu_{t,\theta}\}) - \varphi_{2,\theta}(t, x_{t,\theta}; \{\mu_{t,\theta}\})\Big)dt + \sigma dW_t, \quad x_{s,\theta} = x,$$

where $\dot{\xi}^+_{t,\theta} - \dot{\xi}^-_{t,\theta} = \varphi_\theta(t, x; \{\mu_t\}) = \varphi_{1,\theta}(t, x; \{\mu_{t,\theta}\}) - \varphi_{2,\theta}(t, x; \{\mu_{t,\theta}\})$ is the optimal control function.

Consider the stochastic control problem (Control-FV) with $\{\mu_{t,\theta}\}$ and $v_{T,\infty}(s, x; \{\mu_{t,\theta}\})$ the value function, and let $x_{t,\infty}$ be the optimal controlled process:

$$dx_{t,\infty} = b(x_{t,\infty}, \mu_{t,\infty})dt + \sigma dW_t + d\xi^+_{t,\infty} - d\xi^-_{t,\infty}, \quad x_{s-,\infty} = x.$$

Note that the optimal control $\xi_{t,\infty}$ is of a feedback form. Hence, denote

$$d\varphi_\infty(t, x; \{\mu_{t,\theta}\}) = d\varphi_{1,\infty}(t, x; \{\mu_{t,\theta}\}) - d\varphi_{2,\infty}(t, x; \{\mu_{t,\theta}\}) = d\xi^+_{t,\infty} - d\xi^-_{t,\infty}$$

as the optimal control function for the stochastic control problem (N-FV) with fixed $\{\mu_{t,\theta}\}$.

Denote[3] also

$$\begin{aligned}
dx^i_{t,\theta} &= \Big(b(x^i_{t,\theta}, \mu_{t,\theta}) + \varphi_{1,\theta}(t, x^i_{t,\theta}) - \varphi_{2,\theta}(t, x^i_{t,\theta})\Big)dt + \sigma dW^i_t, \quad x^i_{s,\theta} = x, \\
dx^i_{t,\infty} &= b(x^i_{t,\infty}, \mu_{t,\theta})dt + d\varphi_{1,\infty}(t, x^i_{t,\infty}) - d\varphi_{2,\infty}(t, x^i_{t,\infty}) + \sigma dW^i_t, \quad x^i_{s-,\infty} = x, \\
dx^{i,N}_{t,\theta} &= \Big(\frac{1}{N}\sum_{j=1,\ldots,N} b_0(x^{i,N}_{t,\theta}, x^{i,N}_{t,\theta}) + \varphi_{1,\theta}(t, x^{i,N}_{t,\theta}) - \varphi_{2,\theta}(t, x^{i,N}_{t,\theta})\Big)dt + \sigma dW^i_t, \quad x^{i,N}_{s,\theta} = x,
\end{aligned}$$

Since $(\mu_{t,\theta}, \varphi_\theta)$ is the solution to (MFG-BD) and $x^i_{t,\theta}$ are i.i.d., and $\mu_{t,\theta}$ is the probability measure of $x^i_{t,\theta}$ for any $i = 1, \ldots, N$.

We first establish some technical Lemmas.

[3] In this proof, omit $\{\mu_{t,\theta}\}$ for notations simplicity. Denote $\varphi_{i,\theta}(t, x) = \varphi_{i,\theta}(t, x; \{\mu_{t,\theta}\})$ and $\varphi_{i,\infty}(t, x) = \varphi_{i,\infty}(t, x; \{\mu_{t,\theta}\})$ for $i = 1, 2$.

Lemma 1. *For any* $1 \leq i \leq n$, $\mathbb{E} \sup_{s \leq t \leq T} \left| x^i_{t,\theta} - x^{i,N}_{t,\theta} \right|^2 = O\left(\frac{1}{N}\right)$.

Proof.

$$d(x^i_{t,\theta} - x^{i,N}_{t,\theta}) = \left(\int_{\mathbb{R}} b_0(x^i_{t,\theta}, y)\mu_{t,\theta}(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^{j,N}_{t,\theta}) + \varphi_\theta(t, x^i_{t,\theta}) - \varphi_\theta(t, x^{i,N}_{t,\theta}) \right) dt,$$

and

$$\begin{aligned} d(x^i_{t,\theta} - x^{i,N}_{t,\theta})^2 = \Big\{ 2(x^i_{t,\theta} - x^{i,N}_{t,\theta}) \Big(\int_{\mathbb{R}} b_0(x^i_{t,\theta}, y)\mu_{t,\theta}(dy) \\ - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^{j,N}_{t,\theta}) + \varphi_\theta(t, x^i_{t,\theta}) - \varphi_\theta(t, x^{i,N}_{t,\theta}) \Big) \Big\} dt. \end{aligned}$$

By Assumption (A6), $(x^i_{t,\theta} - x^{i,N}_{t,\theta})\left(\varphi_\theta(t, x^i_{t,\theta}) - \varphi_\theta(t, x^{i,N}_{t,\theta}) \right) \leq 0$. Consequently,

$$\begin{aligned} |x^i_{T,\theta} - x^{i,N}_{T,\theta}|^2 &\leq \int_s^T 2|x^i_{t,\theta} - x^{i,N}_{t,\theta}| \left| \int_{\mathbb{R}} b_0(x^i_{t,\theta}, y)\mu_{t,\theta}(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^{j,N}_{t,\theta}) \right| dt \\ &\leq \int_s^T 2|x^i_{t,\theta} - x^{i,N}_{t,\theta}| \left| \int_{\mathbb{R}} b_0(x^i_{t,\theta}, y)\mu_{t,\theta}(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^{j}_{t,\theta}) \right| dt \\ &\quad + \int_s^T 2|x^i_{t,\theta} - x^{i,N}_{t,\theta}| \left| \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^{j}_{t,\theta}) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^{j,N}_{t,\theta}) \right| dt \\ &\leq \int_s^T 2|x^i_{t,\theta} - x^{i,N}_{t,\theta}| \left| \int_{\mathbb{R}} b_0(x^i_{t,\theta}, y)\mu_{t,\theta}(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^{j}_{t,\theta}) \right| dt \\ &\quad + \int_s^T 2|x^i_{t,\theta} - x^{i,N}_{t,\theta}| \left(\frac{1}{N}\sum_{j=1}^N Lip(b_0)|x^j_{t,\theta} - x^{j,N}_{t,\theta}| \right) dt \\ &\leq \int_s^T |x^i_{t,\theta} - x^{i,N}_{t,\theta}|^2 + \left| \int_{\mathbb{R}} b_0(x^i_{t,\theta}, y)\mu^\theta_t(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^{j}_{t,\theta}) \right|^2 dt \\ &\quad + \int_s^T \frac{1}{N} Lip(b_0) \sum_{j=1}^N \left(|x^i_{t,\theta} - x^{i,N}_{t,\theta}|^2 + |x^j_{t,\theta} - x^{j,N}_{t,\theta}|^2 \right) dt \\ &\leq (1 + Lip(b_0)) \int_s^T |x^i_{t,\theta} - x^{i,N}_{t,\theta}|^2 dt \\ &\quad + \int_s^T \left| \int_{\mathbb{R}} b_0(x^i_{t,\theta}, y)\mu_{t,\theta}(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^{j}_{t,\theta}) \right|^2 dt \\ &\quad + \int_s^T \frac{1}{N} Lip(b_0) \sum_{j=1}^N |x^j_{t,\theta} - x^{j,N}_{t,\theta}|^2 dt. \end{aligned}$$

Hence,

$$
\begin{aligned}
\mathbb{E}|x^i_{T,\theta} - x^{i,N}_{T,\theta}|^2 \leq &(1+Lip(b_0))\mathbb{E}\int_s^T |x^i_{t,\theta} - x^{i,N}_{t,\theta}|^2 dt \\
&+ \mathbb{E}\int_s^T \left| \int_{\mathbb{R}} b_0(x^i_{t,\theta}, y)\mu_{t,\theta}(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^j_{t,\theta}) \right|^2 dt \\
&+ \mathbb{E}\int_s^T Lip(b_0)|x^i_{t,\theta} - x^{i,N}_{t,\theta}|^2 dt,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\left| \int_{\mathbb{R}} b_0(x^i_{t,\theta}, y)\mu_{t,\theta}(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^j_{t,\theta}) \right|^2 \\
&\leq 2\mathbb{E}\left| \int_{\mathbb{R}} b_0(x^i_{t,\theta}, y) - b_0(x^{i,N}_{t,\theta}, y)\mu_t(dy) \right|^2 + 2\mathbb{E}\left| \int_{\mathbb{R}} b_0(x^{i,N}_{t,\theta}, y)\mu_t(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^j_{t,\theta}) \right|^2 \\
&\leq 2\mathbb{E}\left| \int_{\mathbb{R}} Lip(b_0)|x^i_{t,\theta} - x^{i,N}_{t,\theta}|\mu_t(dy) \right|^2 + 2\mathbb{E}\left| \int_{\mathbb{R}} b_0(x^{i,N}_{t,\theta}, y)\mu_t(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^j_{t,\theta}) \right|^2 \\
&= 2Lip(b_0)^2\mathbb{E}|x^i_{t,\theta} - x^{i,N}_{t,\theta}|^2 + \epsilon_N^2,
\end{aligned}
$$

with $\epsilon_N = O\left(\frac{1}{\sqrt{N}}\right)$ from the Central Limit Theorem. Consequently,

$$
\mathbb{E}|x^i_{T,\theta} - x^{i,N}_{T,\theta}|^2 \leq \mathbb{E}\int_s^T (1+2Lip(b_0))|x^i_{t,\theta} - x^{i,N}_{t,\theta}|^2 dt + \mathbb{E}\int_s^T \left(2Lip(b_0)^2|x^i_{t,\theta} - x^{i,N}_{t,\theta}|^2 + \epsilon_N^2\right) dt.
$$

By Gronwall's inequality,

$$
\begin{aligned}
\mathbb{E}\sup_{s\leq t\leq T}|x^i_{T,\theta} - x^{i,N}_{T,\theta}|^2 &\leq \int_s^T \epsilon_N^2 dt \cdot \mathbb{E}\left[\exp(\int_s^T (1+4Lip(b_0)^2)dt)\right] \\
&= \int_s^T \epsilon_N^2 dt \cdot e^{(1+4Lip(b_0)^2)T} = O\left(\frac{1}{N}\right).
\end{aligned}
$$

Therefore, $\mathbb{E}\sup_{s\leq t\leq T}|x^i_{t,\theta} - x^{i,N}_{t,\theta}|^2 = O\left(\frac{1}{N}\right)$. □

Proof of Theorem 3 a). Suppose that the first player chooses a different control $\xi'_t \in \mathcal{U}_\infty$ which is of a bounded velocity and all other players $i = 2, 3, \ldots, N$ choose to stay with the optimal control $\{\xi_{t,\theta}\}$. Denote

$$
d\xi'_t = \dot{\xi}'_t dt = \varphi'(t,x)dt, \qquad \text{and} \qquad d\xi_{t,\theta} = \dot{\xi}_{t,\theta}dt = \varphi_\theta(t,x)dt.
$$

Then the corresponding dynamics for the MFG is

$$
d\tilde{x}^1_{t,\theta} = \left(b(\tilde{x}^1_{t,\theta}, \mu_{t,\theta}) + \varphi'(t, \tilde{x}^1_{t,\theta})\right)dt + \sigma dW^1_t,
$$

and the corresponding dynamics for N-player game are

$$d\tilde{x}^{1,N}_{t,\theta} = \left(\frac{1}{N}\sum_{j=1}^{N} b_0(\tilde{x}^{1,N}_{t,\theta}, \tilde{x}^{j,N}_{t,\theta}) + \varphi'(t, \tilde{x}^{1,N}_{t,\theta})\right) dt + \sigma dW^1_t,$$

$$d\tilde{x}^{i,N}_{t,\theta} = \left(\frac{1}{N}\sum_{j=1}^{N} b(\tilde{x}^{i,N}_{t,\theta}, \tilde{x}^{j,N}_{t,\theta}) + \varphi_\theta(t, \tilde{x}^{i,N}_{t,\theta})\right) dt + \sigma dW^i_t, \qquad 2 \le i \le N.$$

We first show

Lemma 2. $\sup_{2\le i\le N} \mathbb{E} \sup_{0\le t\le T} |x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}| \le O\left(\frac{1}{\sqrt{N}}\right)$.

Proof. For any $2 \le i \le N$,

$$d(x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}) = \left[\frac{1}{N}\sum_{j=1}^{N}\Big(b_0(x^{i,N}_{t,\theta}, x^{j,N}_{t,\theta}) - b_0(\tilde{x}^{i,N}_{t,\theta}, \tilde{x}^{j,N}_{t,\theta})\Big) + \varphi_\theta(t, x^{i,N}_{t,\theta}) - \varphi_\theta(t, \tilde{x}^{i,N}_{t,\theta})\right] dt.$$

Because $\varphi_\theta(t,x)$ is nonincreasing in x,

$$\begin{aligned}
|x^{i,N}_{T,\theta} - \tilde{x}^{i,N}_{T,\theta}|^2 &\le \int_s^T 2(x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta})\left(\frac{1}{N}\sum_{j=1}^{N}\Big(b_0(x^{i,N}_{t,\theta}, x^{j,N}_{t,\theta}) - b_0(\tilde{x}^{i,N}_{t,\theta}, \tilde{x}^{j,N}_{t,\theta})\Big)\right) dt \\
&\le \int_s^T 2(x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta})\frac{1}{N}\sum_{j=1}^{N} Lip(b_0)\Big(|x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}| + |x^{j,N}_{t,\theta} - \tilde{x}^{j,N}_{t,\theta}|\Big)dt \\
&\le 2Lip(b_0)\int_s^T |x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}|^2 + |x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}|\frac{1}{N}\sum_{j=1}^{N}|x^{j,N}_{t,\theta} - \tilde{x}^{j,N}_{t,\theta}|dt \\
&\le 2Lip(b_0)\int_s^T |x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}|^2 + \frac{1}{2N}\sum_{j=1}^{N}\Big(|x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}|^2 + |x^{j,N}_{t,\theta} - \tilde{x}^{j,N}_{t,\theta}|^2\Big)dt \\
&\le Lip(b_0)\int_s^T 3|x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}|^2 + \frac{1}{N}\sum_{j=1}^{N}|x^{j,N}_{t,\theta} - \tilde{x}^{j,N}_{t,\theta}|^2 dt,
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{2\le i\le N}\mathbb{E}\sup_{s\le t\le T}|x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}|^2 \\
&\quad\le Lip(b_0)\int_s^T [\sup_{2\le i\le N}\mathbb{E}\sup_{s\le t'\le t} 3|x^{i,N}_{t',\theta} - \tilde{x}^{i,N}_{t',\theta}|^2 \\
&\qquad + \frac{N-1}{N}\sup_{2\le j\le N}\mathbb{E}\sup_{s\le t'\le t}|x^{j,N}_{t',\theta} - \tilde{x}^{j,N}_{t',\theta}|^2 + \frac{1}{N}\mathbb{E}|x^{1,N}_{t,\theta} - \tilde{x}^{1,N}_{t,\theta}|^2]dt \\
&\quad= Lip(b_0)\int_s^T \left[\frac{4N-1}{N}\sup_{2\le i\le N}\mathbb{E}\sup_{s\le t'\le t}|x^{i,N}_{t',\theta} - \tilde{x}^{i,N}_{t',\theta}|^2 + \frac{1}{N}\mathbb{E}|x^{1,N}_{t,\theta} - \tilde{x}^{1,N}_{t,\theta}|^2\right] dt.
\end{aligned}$$

By Gronwall's inequality,

$$\sup_{2\le i\le N}\mathbb{E}\sup_{s\le t\le T}|x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}|^2 \le Lip(b_0)\int_s^T \frac{1}{N}\mathbb{E}|x^{1,N}_{t,\theta} - \tilde{x}^{1,N}_{t,\theta}|^2 dt \cdot e^{\int_0^T Lip(b_0)\frac{4N-1}{N}dt} = O\left(\frac{1}{N}\right).$$

So, $\sup\limits_{2\leq i\leq N}\mathbb{E}\sup\limits_{s\leq t\leq T}|x_{t,\theta}^{i,N}-\tilde{x}_{t,\theta}^{i,N}|=O\Big(\frac{1}{\sqrt{N}}\Big)$. □

Next, by Lemma 1, for any $2\leq i\leq N$, $\sup\limits_{s\leq t\leq T}\mathbb{E}|x_{t,\theta}^{i}-x_{t,\theta}^{i,N}|=O\left(\frac{1}{\sqrt{N}}\right)$, and by the triangle inequality, $\sup\limits_{2\leq i\leq N}\mathbb{E}\sup\limits_{s\leq t\leq T}|x_{t,\theta}^{i}-\tilde{x}_{t,\theta}^{i,N}|=O(\frac{1}{\sqrt{N}})$. Therefore,

$$\sup_{2\leq i\leq N}\mathbb{E}\sup_{s\leq t\leq T}|x_{t,\theta}^{i}-\tilde{x}_{t,\theta}^{i,N}|+\sup_{1\leq i\leq N}\mathbb{E}\sup_{s\leq t\leq T}|x_{t,\theta}^{i}-x_{t,\theta}^{i,N}|=O\left(\frac{1}{\sqrt{N}}\right).$$

Finally, define

$$d\bar{x}_{t,\theta}^{1,N}=\left(\frac{1}{N}\sum_{j=1}^{N}b_0(\bar{x}_{t,\theta}^{1,N},x_{t,\theta}^{j})+\varphi'(t,\bar{x}_{t,\theta}^{1,N})\right)dt+\sigma dW_t^1,$$

Since $(x-y)(\varphi'(t,x)-\varphi'(t,y))\leq 0$ by Assumption (A6), then a similar proof as that for Lemma 1 shows $\mathbb{E}\sup\limits_{0\leq t\leq T}|\tilde{x}_{t,\theta}^{1,N}-\bar{x}_{t,\theta}^{1,N}|=O\left(\frac{1}{\sqrt{N}}\right)$ and $\mathbb{E}\sup\limits_{0\leq t\leq T}|\bar{x}_{t,\theta}^{1,N}-\tilde{x}_{t,\theta}^{1}|=O\left(\frac{1}{\sqrt{N}}\right)$. Therefore,

$$\begin{aligned}
&J_{T,\theta}^{i,N}(s,x^1,\xi_{\cdot}^{'+},\xi_{\cdot}^{'-};\xi_{\cdot,\theta}^{-1})\\
&=\mathbb{E}\left[\int_s^T\frac{1}{N}\sum_{j=1}^{N}f_0(\tilde{x}_{t,\theta}^{1,N},\tilde{x}_{t,\theta}^{j,N})dt+\gamma_1\varphi_1'(t,\tilde{x}_{t,\theta}^{1,N})-\gamma_2\varphi_2'(t,\tilde{x}_{t,\theta}^{1,N})dt\right]\\
&\geq\mathbb{E}\left[\int_s^T\frac{1}{N}\sum_{j=1}^{N}f_0(\tilde{x}_{t,\theta}^{1,N},x_{t,\theta}^{j})dt+\gamma_1\varphi_1'(t,\tilde{x}_{t,\theta}^{1,N})-\gamma_2\varphi_2'(t,\tilde{x}_{t,\theta}^{1,N})dt\right]-O\left(\frac{1}{\sqrt{N}}\right)\\
&\geq\mathbb{E}\left[\int_s^T\frac{1}{N}\sum_{j=1}^{N}f_0(\bar{x}_{t,\theta}^{1,N},x_{t,\theta}^{j})dt+\gamma_1\varphi_1'(t,\bar{x}_{t,\theta}^{1,N})-\gamma_2\varphi_2'(t,\bar{x}_{t,\theta}^{1,N})dt\right]-O\left(\frac{1}{\sqrt{N}}\right)\\
&\geq\mathbb{E}\left[\int_s^T\int_{\mathbb{R}}f_0(\tilde{x}_{t,\theta}^{1},y)\mu_{t,\theta}(dy)+\gamma_1\varphi_1'(t,\tilde{x}_{t,\theta}^{1})-\gamma_2\varphi_2'(t,\tilde{x}_{t,\theta}^{1})dt\right]-O\left(\frac{1}{\sqrt{N}}\right)\\
&\geq\mathbb{E}\left[\int_s^T\int_{\mathbb{R}}f_0(x_{t,\theta}^{1},y)\mu_{t,\theta}(dy)+\gamma_1\varphi_{1,\theta}(t,x_{t,\theta}^{1})-\gamma_2\varphi_{2,\theta}(t,x_{t,\theta}^{1})dt\right]-O\left(\frac{1}{\sqrt{N}}\right)\\
&=\mathbb{E}\left[\int_s^T\frac{1}{N}\sum_{j=1}^{N}f_0(x_{t,\theta}^{1,N},x_{t,\theta}^{j,N})dt+\gamma_1\varphi_{1,\theta}(t,x_{t,\theta}^{1,N})-\gamma_2\varphi_{2,\theta}(t,x_{t,\theta}^{1,N})dt\right]-O\left(\frac{1}{\sqrt{N}}\right)\\
&=J_{T,\theta}^{i,N}(s,x^1,\xi_{\cdot,\theta}^{+},\xi_{\cdot,\theta}^{-};\xi_{\cdot,\theta}^{-1})-O\left(\frac{1}{\sqrt{N}}\right),
\end{aligned}$$

the last inequality is due to the optimality of φ for problem (MFG-BD), and the last equality follows from the Central Limit Theorem. □

Proof of Theorem 3 b). Let all players except player one choose the optimal controls $(\xi_{\cdot,\theta}^{+},\xi_{\cdot,\theta}^{-})$, let player one choose any other controls $(\xi_{\cdot,\theta}^{'+},\xi_{\cdot,\theta}^{'-})\in\mathcal{U}_{T,\infty}$. Denote

$$d\xi_t'=d\varphi'(t,x)=d\varphi_1'(t,x)-d\varphi_2'(t,x),$$

$$
\begin{aligned}
d\tilde{x}^1_{t,\infty} &= b(\tilde{x}^1_{t,\infty}, \mu^\theta_t)dt + d\varphi'_1(t, \tilde{x}^1_{t,\infty}) - d\varphi'_2(t, \tilde{x}^1_{t,\infty}) + \sigma dW^1_t \quad \tilde{x}^1_{s-,\infty} = x,\\
d\tilde{x}^{1,N}_{t,\infty} &= \frac{1}{N}\sum_{j=1,\ldots,N} b_0(\tilde{x}^{1,N}_{t,\infty}, \tilde{x}^{j,N}_{t,\infty})dt + d\varphi'_1(t, \tilde{x}^{1,N}_{t,\infty}) - d\varphi'_2(t, \tilde{x}^{1,N}_{t,\infty}) + \sigma dW^1_t, \quad \tilde{x}^{1,N}_{s-,\infty} = x,\\
d\tilde{x}^{i,N}_{t,\infty} &= \left(\frac{1}{N}\sum_{j=1,\ldots,N} b_0(\tilde{x}^{i,N}_{t,\infty}, \tilde{x}^{j,N}_{t,\infty}) + \varphi_{1,\theta}(t, \tilde{x}^{i,N}_{t,\infty}) - \varphi_{2,\theta}(t, \tilde{x}^{i,N}_{t,\infty})\right)dt + \sigma dW^i_t, \quad x^{i,N}_{s-,\infty} = x,\\
&\quad \text{for } i = 2, \ldots, N.
\end{aligned}
$$

Then,

$$
d(x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\infty}) = \left[\frac{1}{N}\sum_{j=1}^N \left(b_0(x^{i,N}_{t,\theta}, x^{j,N}_{t,\theta}) - b_0(\tilde{x}^{i,N}_{t,\infty}, \tilde{x}^{j,N}_{t,\infty})\right) + \varphi_\theta(t, x^{i,N}_{t,\theta}) - \varphi_\theta(t, \tilde{x}^{i,N}_{t,\infty})\right] dt.
$$

By definition, $\varphi_\theta(t,x)$ is nonincreasing in x. Hence, a similar proof to the one for Lemma 2 yields

Lemma 3. $\sup_{2\le i\le N} \mathbb{E} \sup_{s\le t\le T} |x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\infty}| = O\left(\frac{1}{\sqrt{N}}\right)$.

From Lemma 1 and the triangle inequality, $\sup_{2\le i\le N} \mathbb{E} \sup_{s\le t\le T} |x^i_{t,\theta} - \tilde{x}^{i,N}_{t,\infty}| = O\left(\frac{1}{\sqrt{N}}\right)$. Therefore,

$$
\sup_{2\le i\le N} \mathbb{E} \sup_{s\le t\le T} |x^i_{t,\theta} - \tilde{x}^{i,N}_{t,\infty}| + \sup_{1\le i\le N} \mathbb{E} \sup_{s\le t\le T} |x^i_{t,\theta} - x^{i,N}_{t,\theta}| = O\left(\frac{1}{\sqrt{N}}\right).
$$

Since $d\varphi'(t,x)$ is also nonincreasing in x, then again the same proof as that for Lemma 1 shows

$$
\mathbb{E} \sup_{s\le t\le T} |\tilde{x}^{1,N}_{t,\infty} - \tilde{x}^1_{t,\infty}| = O\left(\frac{1}{\sqrt{N}}\right).
$$

By the Lipschitz continuity of f, f_0,

$$
\begin{aligned}
&J^{1,N}_{T,\infty}(s, x, \xi'^{+}_{\cdot}, \xi'^{-}_{\cdot}; \xi^{-1}_{\cdot,\theta})\\
&= \mathbb{E}\left[\int_s^T \frac{1}{N}\sum_{j=1}^N f_0(\tilde{x}^{1,N}_{t,\infty}, \tilde{x}^{j,N}_{t,\infty})dt + \gamma_1 d\varphi'_1(t, \tilde{x}^{1,N}_{t,\infty}) + \gamma_2 d\varphi'_2(t, \tilde{x}^{1,N}_{t,\infty})\right]\\
&\ge \mathbb{E}\left[\int_s^T \frac{1}{N}\sum_{j=1}^N f_0(\tilde{x}^{1,N}_{t,\infty}, x^j_{t,\theta})dt + \gamma_1 d\varphi'_1(t, \tilde{x}^{1,N}_{t,\infty}) + \gamma_2 d\varphi'_2(t, \tilde{x}^{1,N}_{t,\infty})\right] - O\left(\frac{1}{\sqrt{N}}\right)\\
&\ge \mathbb{E}\left[\int_s^T \int_{\mathbb{R}} f(\tilde{x}^{1,N}_{t,\infty}, y)\mu_{t,\theta}(dy)dt + \gamma_1 d\varphi'_1(t, \tilde{x}^{1,N}_{t,\infty}) + \gamma_2 d\varphi'_2(t, \tilde{x}^{1,N}_{t,\infty})\right] - O\left(\frac{1}{\sqrt{N}}\right)\\
&\ge \mathbb{E}\left[\int_s^T \int_{\mathbb{R}} f(\tilde{x}^1_{t,\infty}, y)\mu_{t,\theta}(dy)dt + \gamma_1 d\varphi'_1(t, \tilde{x}^{1,N}_{t,\infty}) + \gamma_2 d\varphi'_2(t, \tilde{x}^{1,N}_{t,\infty})\right] - O\left(\frac{1}{\sqrt{N}}\right).
\end{aligned}
$$

By definition of φ'_1, φ'_2,

$$
\begin{aligned}
&\mathbb{E}\left|d\varphi'_1(t, \tilde{x}^{1,N}_{t,\infty}) - d\varphi'_1(t, \tilde{x}^1_{t,\infty}) - d\varphi'_2(t, \tilde{x}^{1,N}_{t,\infty}) + d\varphi'_2(t, \tilde{x}^1_{t,\infty})\right|\\
&\quad \le \mathbb{E}d|\tilde{x}^{1,N}_{t,\infty} - \tilde{x}^1_{t,\infty}| + \mathbb{E}\left|\frac{1}{N}\sum_{j=1,\ldots,N} b_0(\tilde{x}^{1,N}_{t,\infty}, \tilde{x}^{j,N}_{t,\infty}) - b(\tilde{x}^{\infty 1}_t, \mu_{t,\theta})\right| dt = O\left(\frac{1}{\sqrt{N}}\right),
\end{aligned}
$$

and by definition of φ'_1, φ'_2,

$$\begin{aligned}
&\left|\left(d\varphi'_1(t,\tilde{x}^{1,N}_{t,\infty}) - d\varphi'_1(t,\tilde{x}^{1}_{t,\infty})\right) + \left(-d\varphi'_2(t,\tilde{x}^{1,N}_{t,\infty}) + d\varphi'_2(t,\tilde{x}^{1}_{t,\infty})\right)\right| \\
&= \left|d\varphi'_1(t,\tilde{x}^{1,N}_{t,\infty}) - d\varphi'_1(t,\tilde{x}^{1}_{t,\infty})\right| + \left|-d\varphi'_2(t,\tilde{x}^{1,N}_{t,\infty}) + d\varphi'_2(t,\tilde{x}^{1}_{t,\infty})\right|.
\end{aligned}$$

Therefore,

$$\mathbb{E}\sup_{s\le t\le T}\left|d\varphi'_1(t,\tilde{x}^{1,N}_{t,\infty}) - d\varphi'_1(t,\tilde{x}^{1}_{t,\infty})\right| = O\left(\frac{1}{\sqrt{N}}\right),$$

$$\mathbb{E}\sup_{s\le t\le T}\left|-d\varphi'_2(t,\tilde{x}^{1,N}_{t,\infty}) + d\varphi'_2(t,\tilde{x}^{1}_{t,\infty})\right| = O\left(\frac{1}{\sqrt{N}}\right),$$

and

$$\begin{aligned}
&\mathbb{E}\left[\int_s^T\int_{\mathbb{R}} f(\tilde{x}^1_{t,\infty},y)\mu_{t,\theta}(dy)dt + \gamma_1 d\varphi'_1(t,\tilde{x}^{1,N}_{t,\infty}) + \gamma_2 d\varphi'_2(t,\tilde{x}^{1,N}_{t,\infty})\right] - O\left(\frac{1}{\sqrt{N}}\right) \\
&\ge \mathbb{E}\left[\int_s^T\int_{\mathbb{R}} f(\tilde{x}^1_{t,\infty},y)\mu_{t,\theta}(dy)dt + \gamma_1 d\varphi'_1(t,\tilde{x}^{1}_{t,\infty}) + \gamma_2 d\varphi'_2(t,\tilde{x}^{1}_{t,\infty})\right] - O\left(\frac{1}{\sqrt{N}}\right) \\
&\ge \mathbb{E}\left[\int_s^T\int_{\mathbb{R}} f(x^1_{t,\infty},y)\mu_{t,\theta}(dy)dt + \gamma_1 d\varphi_{1,\infty}(t,x^{1}_{t,\infty}) + \gamma_2 d\varphi_{2,\infty}(t,x^{1}_{t,\infty})\right] - O\left(\frac{1}{\sqrt{N}}\right) \\
&= v_{T,\infty}(s,x;\{\mu_{t,\theta}\}) - O\left(\frac{1}{\sqrt{N}}\right).
\end{aligned}$$

The last inequality is due to the optimality of φ_∞.

Now, by Theorem 1,

$$\left|v_{T,\theta}(s,x:\{\mu_{t,\theta}\})) - v_{T,\infty}(s,x;\{\mu_{t,\theta}\}))\right| \le \epsilon_\theta.$$

Hence, by $\mathbb{E}\sup_{s\le t\le T}|x^i_{t,\theta} - x^{i,N}_{t,\theta}| = \epsilon_N$ and by the analysis as in the previous steps

$$\begin{aligned}
&J^{1,N}_{T,\infty}(s,x,\xi'^{+}_{\cdot},\xi'^{-}_{\cdot};\xi^{-1}_{\cdot,\theta}) = v_{T,\infty}(s,x;\{\mu_{t,\theta}\}) - \epsilon_N \\
&\ge v_{T,\theta}(s,x;\{\mu_{t,\theta}\}) - (\epsilon_N + \epsilon_\theta) \\
&= \mathbb{E}\left[\int_s^T\int_{\mathbb{R}} f(x^1_{t,\theta},y)\mu_{t,\theta}(dy)dt + \gamma_1 d\varphi_{1,\theta}(t,x^1_{t,\theta}) + \gamma_2 d\varphi_{2,\theta}(t,x^1_{t,\theta})\right] - (\epsilon_N + \epsilon_\theta) \\
&\ge \mathbb{E}\left[\int_s^T \frac{1}{N}\sum_{j=1}^N f_0(x^{1,N}_{t,\theta},x^{j,N}_{t,\theta})dt + \gamma_1 d\varphi_{1,\theta}(t,x^{1,N}_{t,\theta}) + \gamma_2 d\varphi_{2,\theta}(t,x^{1,N}_{t,\theta})\right] - (\epsilon_N + \epsilon_\theta) \\
&= J^{1,N}_{T,\infty}(s,x,\xi^{+}_{\cdot,\theta},\xi^{-}_{\cdot,\theta};\xi^{-1}_{\cdot,\theta}) - (\epsilon_N + \epsilon_\theta). \qquad \square
\end{aligned}$$

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