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Optimal partially reversible investment with entry decision and general production function

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Abstract

This paper studies the problem of a company that adjusts its stochastic production capacity in reversible investments with controls of expansion and contraction. The company may also decide on the activation time of its production. The profit production function is of a very general form satisfying minimal standard assumptions. The objective of the company is to find an optimal entry and production decision to maximize its expected total net profit over an infinite time horizon. The resulting dynamic programming principle is a two-step formulation of a singular stochastic control problem and an optimal stopping problem. The analysis of value functions relies on viscosity solutions of the associated Bellman variational inequations. We first state several general properties and in particular smoothness results on the value functions. We then provide a complete solution with explicit expressions of the value functions and the optimal controls: the company activates its production once a fixed entry-threshold of the capacity is reached, and invests in capital so as to maintain its capacity in a closed bounded interval. The boundaries of these regions can be computed explicitly and their behavior is studied in terms of the parameters of the model.

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1. Introduction

Consider the following model. A company wishes to adjust its production capacity according to market fluctuations. Specifically, the company is given the option to invest in producing a single commodity. The company is free to choose the exact time at which the production would be activated. Activating a production incurs a cost, and the production itself yields a profit which is the function of the product capacity. The objective of the company is to find the optimal investment and entry time decision in order to maximize its overall expected net profit. An extensive review of such problems can be found in the book by Dixit and Pindyck [5].

The underlying motivation to study such investment models is to offer companies some flexibility in their production levels according to market demand fluctuations. Ideally, when the market improves and the demand increases, companies should be able to expand their production levels. Similarly, when the market falls, companies should be able to reduce production or by abandon the production. Regrettably, very few models have explicitly incorporated the two important factors in this scenario—*expansion* and *contraction*! In many existing models, the possibility of varying capacity was captured either by adding choices of entry and exit times (as in [6]) or by focusing on the possibility of only purchasing capital (irreversible investment), an unreasonable simplification. For instance, based on the model of Pindyck [13], Oksendal [12] considers the problem of a company which expands its capacity in irreversible investment over an infinite horizon. Recently, Chiarolla and Haussmann [3] have studied an irreversible investment model in a finite time horizon, without entry decision, by introducing the singular control on the capital expansion. Their approach is to solve the singular control problem by studying an associated optimal stopping problem; they use a verification theorem argument and obtain an explicit solution only for the special case of the power production function.

We propose a general investment model that incorporates expansion and contraction according to the market change, in addition to the entry decision of activating production. Therefore, the overall model of the company is the same as stated earlier, except that once the production is activated, the company can adjust its capital level by proper controls of expansion and contraction, reflecting a partially reversible investment. The net profit of such an investment depends on the running production function of the actual capacity, the profit of contraction (e.g. via spinning off part of the business), and the cost of expanding the capital. The production function is of a very general form, satisfying minimal standard assumptions; it includes the special case of power or Cobb–Douglas functions usually considered in many investment models. The expansion and contraction decisions are modeled by a pair of singular controls. The company's objective is to maximize the expected profit over an infinite time horizon, with choices of the entry time and controls of expansion and contraction.

Using the dynamic programming principle, we reduce the original control problem into a two-stage procedure. First, we introduce an auxiliary singular stochastic control problem corresponding to an immediate entry decision. The value function of the original problem is then formulated as an optimal stopping time problem on the entry decision, with payoff function equal to the auxiliary value function. The two value functions are analyzed based on viscosity solutions of the associated Bellman variational inequalities. We first derive general properties and, in particular, regularity results on the value functions. We then provide a complete and explicit solution to the value functions and the optimal policies: the company activates its production once a fixed entry-threshold of the capacity is reached, and invests in capital so as to maintain its capacity in a closed bounded interval. The boundaries of these regions can be computed quite explicitly and their behavior is discussed in terms of the parameters of the model.

From a mathematical viewpoint, we make extensive use of viscosity solutions approach. This allows us to go beyond the classical approach on optimal investment models where the principal effort is to first construct (by ad hoc methods) a solution to the Bellman equation, and then validate the optimality of the solution by a sufficient verification theorem for smooth functions. Explicit solutions to the associated Bellman equation may then be derived only for special cases, typically for power or logarithmic profit functions. We, on the other hand, start by studying and deriving the general properties on the value functions via the dynamic programming principle and viscosity arguments. Using the concavity property of the auxiliary value function, we prove that it satisfies necessarily the Hamilton–Jacobi–Bellman (HJB) equation in a classical C^2 sense. (A similar approach can be found in the papers by Shreve and Soner [16] and Choulli et al. [4].) Moreover, it appears that the value function for the optimal stopping problem is not concave in general. However, we are able to prove the smooth-fit condition, i.e., the continuous differentiability C^1 of this value function. From a detailed analysis, we explicitly solve the two control problems and construct the optimal controls—the entry decision and the expansion and contraction policies.

The rest of the paper goes as follows. In Section 2, we give a mathematical formulation of the problem. In Section 3, we show how the problem can be reduced into a two-stage procedure by solving first an auxiliary singular control problem and then a related optimal stopping problem. We analyze and derive some general properties of the auxiliary value function in Section 4. Using viscosity solutions arguments, we state in Section 5 the C^2 smoothness of this value function that satisfies the associated Bellman equation in a classical sense. Section 6 is devoted to the explicit construction of the solution to the auxiliary singular control problem. In Section 7, we return to the original problem by proving the C^1 smoothness of the value function and explicitly solving the associated optimal stopping problem. As a by-product, we give a construction of the optimal entry decision and the optimal expansion and contraction controls. Finally, Section 8 gives some economic interpretations of our mathematical results, along with concluding remarks.

2. Formulation of the problem

We consider a company producing a single commodity. In the absence of intervention and control, the production capacity K_t evolves according to

$$dK_t = K_t(\delta\, dt + \gamma\, dW_t). \tag{2.1}$$

The process W is a standard Brownian on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geqslant 0}, P)$. We assume that $\mathbb{F}$ is the augmented filtration of the σ-algebra generated by W. δ is the appreciation (when $\delta \geqslant 0$) or depreciation (when $\delta < 0$) rate of the production capacity, and $\gamma > 0$ represents the volatility of the capital stock. We denote by $\mathscr{T}$ the set of all $\mathbb{F}$-stopping times.

The company's decision for production is made at some stopping time $\tau_I \in \mathscr{T}$ and incurs a fixed cost $C_I \geqslant 0$. From that time, the company can increase its capital level. We assume that increased capacity will be converted to p units of investment cost. In order to allow no-arbitrage, we assume that the reduction of investment generates a profit with conversion factor $(1-\lambda)p$, where $\lambda \in (0,1)$. The possibility for the company to reduce capital reflects partial reversibility. The production process, which is the control of investment, is then described by a pair $(L, M) \in \mathscr{P}(\tau_I)$, a set of right-continuous with left-hand limits adapted processes, nonnegative and nondecreasing, with $L_t = M_t = 0$ for $t < \tau_I$. Hence, L_t and M_t, respectively, represent the cumulative expansion and reduction of capital until time t, once the production is active.

Given an initial capital $k \geqslant 0$, and controls $\tau_I \in \mathscr{T}$, $(L, M) \in \mathscr{P}(\tau_I)$, the company's production capacity evolves according to

$$\mathrm{d}K_t = K_t(\delta\,\mathrm{d}t + \gamma\,\mathrm{d}W_t) + \mathrm{d}L_t - \mathrm{d}M_t, \quad K_{0^-} = k. \tag{2.2}$$

Given the initial capital k and the entry decision $\tau_I \in \mathscr{T}$, we say that the policy $(L, M) \in \mathscr{P}(\tau_I)$ is *admissible* if the nonbankruptcy constraint

$$K_t \geqslant 0, \quad t \geqslant 0 \tag{2.3}$$

is satisfied and if the integrability condition

$$E\left[\int_0^\infty \mathrm{e}^{-rt}\,\mathrm{d}L_t\right] < \infty \tag{2.4}$$

holds, where $r > 0$ is a fixed positive discount factor. We denote by $\mathscr{A}_{\tau_I}(k)$ the set of all such admissible policies (L, M). This set is clearly nonempty since it contains the zero control $L = M = 0$.

The instantaneous operating profit of the company is a function $\Pi(K_t)$ of the production capacity. The production profit function Π is assumed to be continuous on $\mathbb{R}_+$, nondecreasing, concave on $(0, \infty)$, with $\Pi(0) = 0$. And we denote by Π^{-1} the inverse of Π:

$$\Pi^{-1}(c) = \inf\{k > 0 : \Pi(k) \geqslant c\}, \quad c > 0.$$

Moreover, we impose two standing assumptions on Π.

A1. Π satisfies the Inada condition at 0, i.e.,

$$\lim_{k \downarrow 0} \frac{\Pi(k)}{k} = \infty. \tag{2.5}$$

A2. The Fenchel–Legendre transform of Π is finite on $(0,\infty)$, i.e.,

$$\tilde{\Pi}(z) = \sup_{k>0}[\Pi(k) - kz] < \infty \quad \forall z > 0. \tag{2.6}$$

A typical example is the Cobb–Douglas production function which leads to a profit function of the form

$$\Pi(k) = Ck^{\alpha} \quad \text{with } C > 0, \quad 0 < \alpha < 1. \tag{2.7}$$

The company's objective is then to maximize the expected profit over an infinite time horizon:

$$J(k, L, M, \tau_I) = E\left[\int_{\tau_I}^{\infty} \mathrm{e}^{-rt}(\Pi(K_t)\,\mathrm{d}t - p\,\mathrm{d}L_t + (1-\lambda)p\,\mathrm{d}M_t) - \mathrm{e}^{-r\tau_I}C_I\right],$$

over all $\tau_I \in \mathscr{T}$ and admissible policies $(L, M) \in \mathscr{A}_{\tau_I}(k)$. Note that $J(k, L, M, \tau_I)$ is well defined, valued in $(-\infty, \infty]$. Accordingly, the value function is defined as

$$v(k) = \sup_{\substack{\tau_I \in \mathscr{T}, \\ (L,M) \in \mathscr{A}_{\tau_I}(k)}} J(k, L, M, \tau_I), \quad k \geqslant 0. \tag{2.8}$$

Finally, for ease of exposition, we will assume

A3. $r > \delta$.

We will see later in Lemma 4.2 that this last condition, together with A2, ensures that the value function v is finite. This assumption makes economic sense in that most investment models actually assume $\delta < 0$, $r > 0$ which naturally satisfies A3.

Moreover, one can verify that when $r < \delta$, $v(k)$ is infinite for all $k > 0$. Indeed, by considering for any $n \in \mathbb{N}$, the decision $\tau_I = 0$, and the admissible policy (L^n, M^n) defined by $L^n = 0$ and $M_t^n = 0$, $t < n$, $M_t^n = K_n$, $t \geqslant n$, we have

$$J(k, 0, M^n, 0) \geqslant E[\mathrm{e}^{-rn}p(1-\lambda)K_n - C_I] = kp(1-\lambda)\mathrm{e}^{(\delta-r)n} - C_I.$$

Sending n to infinity shows that $v(k) = \infty$.

3. Auxiliary optimization problem

Let us first consider the auxiliary value function associated with the optimal problem when the decision for producing is made immediately, namely $\tau_I = 0$:

$$\hat{v}(k) = \sup_{(L,M) \in \mathscr{A}_0(k)} \hat{J}(k, L, M), \quad k \geqslant 0, \tag{3.1}$$

where

$$\hat{J}(k, L, M) = J(k, L, M, 0) + C_I \\ = E\left[\int_0^\infty e^{-rt}(\Pi(K_t)\,dt - p\,dL_t + (1-\lambda)p\,dM_t)\right].$$

By taking $L = M = 0$, we immediately see that $\hat{v}$ is nonnegative.

Problem (3.1) is a singular control problem. By the definition of $\hat{v}$, we have, for all $(L, M) \in \mathscr{A}_0(k)$

$$\hat{v}(k) \geqslant E\left[\int_0^\infty e^{-rt}(\Pi(K_t)\,dt - p\,dL_t + (1-\lambda)p\,dM_t)\right].$$

Moreover, for all $\varepsilon > 0$, one can find $(L^\varepsilon, M^\varepsilon) \in \mathscr{A}_0(k)$ such that

$$\hat{v}(k) \leqslant E\left[\int_0^\infty e^{-rt}(\Pi(K_t)\,dt - p\,dL_t^\varepsilon + (1-\lambda)p\,dM_t^\varepsilon)\right] + \varepsilon.$$

In fact, one can replace the initial time 0 in the above relations by an arbitrary stopping time $\tau_I \in \mathscr{T}$. More precisely, let $k > 0$ and $\tau_I \in \mathscr{T}$. Then, for all $(L, M) \in \mathscr{P}(\tau_I) \cap \mathscr{A}_0(k)$, we have

$$e^{-r\tau_I}\hat{v}(K_{\tau_I^-}) \geqslant E\left[\int_{\tau_I}^\infty e^{-rt}(\Pi(K_t)\,dt - p\,dL_t + (1-\lambda)p\,dM_t)\middle|\mathscr{F}_{\tau_I}\right]. \tag{3.2}$$

Moreover, for all $\varepsilon > 0$, one can find $(L^\varepsilon, M^\varepsilon) \in \mathscr{P}(\tau_I) \cap \mathscr{A}_0(k)$ such that

$$e^{-r\tau_I}\hat{v}(K_{\tau_I^-}) \leqslant E\left[\int_{\tau_I}^\infty e^{-rt}(\Pi(K_t)\,dt - p\,dL_t^\varepsilon + (1-\lambda)p\,dM_t^\varepsilon)\middle|\mathscr{F}_{\tau_I}\right] + \varepsilon. \tag{3.3}$$

The details of this replacement involve measurability issues that are quite technical and will be addressed later. Instead, the validity of the two inequalities (3.2) and (3.3) will be justified a posteriori in Section 7.

Let us denote the capital in the absence of trading by $\hat{K}$. Then,

$$d\hat{K}_t = \hat{K}_t(\delta\,dt + \gamma\,dW_t), \quad \hat{K}_0 = k. \tag{3.4}$$

Note that for all $\tau_I \in \mathscr{T}$, $(L, M) \in \mathscr{P}(\tau_I)$, $K_{\tau_I^-} = \hat{K}_{\tau_I}$. Consider the optimal stopping problem

$$w(k) = \sup_{\tau_I \in \mathscr{T}} E[e^{-r\tau_I}(\hat{v}(\hat{K}_{\tau_I}) - C_I)]. \tag{3.5}$$

Then, from (3.2), we get $w \geqslant v$. Moreover, assuming (3.3), we would have $w(k) \leqslant v(k) + \varepsilon$, for all $\varepsilon > 0$. Therefore $w = v$.

The equality $v = w$ has not been rigorously proved yet, but will be proved later in Theorem 7.3. As a by-product, we will also construct the optimal entry decision and optimal policies. Hence, we decompose the control problem (2.8) into the successive resolution of a singular control problem (3.1) and then of an optimal stopping problem (3.5).

The HJB equation associated with the singular control problem (3.1) is

$$\min\{r\hat{v} - \mathscr{L}\hat{v} - \Pi, -\hat{v}' + p, \hat{v}' - (1-\lambda)p\} = 0, \quad \text{on } (0,\infty), \tag{3.6}$$

while the HJB equation associated with the optimal stopping problem (3.5) is

$$\min\{rw - \mathscr{L}w, w + C_I - \hat{v}\} = 0, \quad \text{on } (0,\infty). \tag{3.7}$$

Here $\mathscr{L}$ is the second-order operator associated to the diffusion system $\hat{K}$, i.e., for any C^2 function φ:

$$\mathscr{L}\varphi = \tfrac{1}{2}\gamma^2 k^2 \varphi'' + \delta k \varphi'.$$

4. Some properties on the auxiliary value function

We first state a standard comparison theorem, which says that any smooth function, which is a supersolution to the Bellman equation (3.6), is a majorant of $\hat{v}$.

To this end, first recall in our context Itô's formula for cadlag semimartingales (cf. [11]).

Let $\varphi \in C^2(0,\infty)$, $\tau_1 \leqslant \tau_2$ be a.s. finite stopping times, $k \geqslant 0$ and $(L,M) \in \mathscr{A}_{\tau_1}(k)$. Then, we have

$$\begin{aligned} e^{-r\tau_2}\varphi(K_{\tau_2}) = {} & e^{-r\tau_1}\varphi(K_{\tau_1^-}) + \int_{\tau_1}^{\tau_2} e^{-rt}(-r\varphi + \mathscr{L}\varphi)(K_t)\,dt \\ & + \int_{\tau_1}^{\tau_2} e^{-rt}\gamma K_t \varphi'(K_t)\,dW_t + \int_{\tau_1}^{\tau_2} e^{-rt}\varphi'(K_t)(\,dL_t^c - dM_t^c) \\ & + \sum_{\tau_1 \leqslant t \leqslant \tau_2} e^{-rt}[\varphi(K_t) - \varphi(K_{t^-})], \end{aligned} \tag{4.1}$$

where

$$L_t^c = L_t - \sum_{0 \leqslant s \leqslant t} \Delta L_s, \quad \Delta L_t = L_t - L_{t^-},$$

$$M_t^c = M_t - \sum_{0 \leqslant s \leqslant t} \Delta M_s, \quad \Delta M_t = M_t - M_{t^-}$$

are the continuous and discontinuous parts of L and M.

Proposition 4.1. *Let φ be a nonnegative C^2 function, supersolution on $(0,\infty)$ to (3.6), i.e.,*

$$\min\{r\varphi - \mathscr{L}\varphi - \Pi(k), -\varphi' + p, \varphi' - (1-\lambda)p\} \geqslant 0, \quad k > 0. \tag{4.2}$$

Then,

$$\hat{v}(k) \leqslant \varphi(k) \quad \forall k > 0.$$

Proof. For $(L,M) \in \mathscr{A}_0(k)$, set $\tau_n = \inf\{t \geqslant 0 : K_t \geqslant n\} \wedge n$, $n \in \mathbb{N}$, and apply Itô's formula (4.1) between the a.s. finite stopping times 0 and τ_n. Then, taking expectation and noting that the integrand in the stochastic integral is bounded on

$[0, \tau_n]$, we get

$$E[\mathrm{e}^{-r\tau_n}\varphi(K_{\tau_n})] = \varphi(k) + E\left[\int_0^{\tau_n} \mathrm{e}^{-rt}(-r\varphi + \mathscr{L}\varphi)(K_t)\,\mathrm{d}t\right] + E\left[\int_0^{\tau_n} \mathrm{e}^{-rt}\varphi'(K_t)(\mathrm{d}L_t^{\mathrm{c}} - \mathrm{d}M_t^{\mathrm{c}})\right] + E\left[\sum_{0 \leqslant t \leqslant \tau_n} \mathrm{e}^{-rt}[\varphi(K_t) - \varphi(K_{t^-})]\right]. \tag{4.3}$$

Since $p(1-\lambda) \leqslant \varphi' \leqslant p$, and $K_t - K_{t^-} = \Delta L_t - \Delta M_t$, the mean-value theorem implies that

$$\varphi(K_t) - \varphi(K_{t^-}) \leqslant p\,\Delta L_t - p(1-\lambda)\Delta M_t.$$

Using again the inequality $p(1-\lambda) \leqslant \varphi' \leqslant p$ in the integrals in $\mathrm{d}L^{\mathrm{c}} - \mathrm{d}M^{\mathrm{c}}$ in (4.3), and recalling $-r\varphi + \mathscr{L}\varphi \leqslant -\Pi$, we then obtain

$$\begin{aligned} E[\mathrm{e}^{-r\tau_n}\varphi(K_{\tau_n})] &\leqslant \varphi(k) - E\left[\int_0^{\tau_n} \mathrm{e}^{-rt}\Pi(K_t)\,\mathrm{d}t\right] \\ &\quad + E\left[\int_0^{\tau_n} \mathrm{e}^{-rt}(p\,\mathrm{d}L_t^{\mathrm{c}} - p(1-\lambda)\,\mathrm{d}M_t^{\mathrm{c}})\right] \\ &\quad + E\left[\sum_{0 \leqslant t \leqslant \tau_n} \mathrm{e}^{-rt}(p\,\Delta L_t - p(1-\lambda)\,\Delta M_t)\right] \\ &= \varphi(k) - E\left[\int_0^{\tau_n} \mathrm{e}^{-rt}\Pi(K_t)\,\mathrm{d}t\right] \\ &\quad + E\left[\int_0^{\tau_n} \mathrm{e}^{-rt}(p\,\mathrm{d}L_t - p(1-\lambda)\,\mathrm{d}M_t)\right]. \end{aligned}$$

Therefore,

$$E\left[\int_0^{\tau_n} \mathrm{e}^{-rt}(\Pi(K_t)\,\mathrm{d}t - p\,\mathrm{d}L_t + p(1-\lambda)\,\mathrm{d}M_t)\right] + E[\mathrm{e}^{-r\tau_n}\varphi(K_{\tau_n})] \leqslant \varphi(k).$$

Since φ is nonnegative, we then get

$$\varphi(k) \geqslant E\left[\int_0^{\tau_n} \mathrm{e}^{-rt}(\Pi(K_t)\,\mathrm{d}t + p(1-\lambda)\,\mathrm{d}M_t)\right] - E\left[\int_0^{\infty} \mathrm{e}^{-rt}p\,\mathrm{d}L_t\right].$$

Applying Fatou's lemma, by taking $n \to \infty$ in the last inequality, yields

$$E\left[\int_0^{\infty} \mathrm{e}^{-rt}(\Pi(K_t)\,\mathrm{d}t - p\,\mathrm{d}L_t + p(1-\lambda)\,\mathrm{d}M_t)\right] \leqslant \varphi(k).$$

Now, $\hat{v}(k) \leqslant \varphi(k)$ follows immediately from the arbitrariness of (L, M). $\square$

We now list a few properties on the value function $\hat{v}$.

Lemma 4.1. *For all $k \geqslant 0$ and $l, m \geqslant 0$ such that $k + l - m \geqslant 0$, we have*

$$\hat{v}(k) \geqslant -pl + p(1-\lambda)m + \hat{v}(k + l - m). \tag{4.4}$$

Proof. For any $(L,M) \in \mathscr{A}_0(k+l-m)$, consider the control $(\tilde{L},\tilde{M})$ defined by $\tilde{L}_{0^-} = \tilde{M}_{0^-} = 0$, and $\tilde{L}_t = L_t + l$, $\tilde{M}_t = M_t + m$, for $t \geqslant 0$. Let $\tilde{K}$ be the solution of (2.2) with control $(\tilde{L},\tilde{M})$ and initial condition $\tilde{K}_{0^-} = k$. Then, $\tilde{K}_t = K_t + l - m$ for $t \geqslant 0$, and $(\tilde{L},\tilde{M}) \in \mathscr{A}_0(k)$. We thus get

$$\begin{aligned}\hat{v}(k) &\geqslant \hat{J}(k,\tilde{L},\tilde{M}) = E\left[\int_0^\infty e^{-rt}(\Pi(\tilde{K}_t)\,dt - p\,d\tilde{L}_t + (1-\lambda)p\,d\tilde{M}_t)\right] \\ &= \hat{J}(k+l-m,L,M) - pl + (1-\lambda)pm.\end{aligned}$$

We obtain the required result from the arbitrariness of (L,M). □

Furthermore, recalling assumption A2 on Π, we obtain

Lemma 4.2. *For any* $q \in [(1-\lambda)p, p]$,

$$kp(1-\lambda) \leqslant \hat{v}(k) \leqslant \frac{\tilde{\Pi}((r-\delta)q)}{r} + kq, \quad k>0. \tag{4.5}$$

Proof. (a) The lower bound immediately follows from (4.4) by taking $l=0$, $m=k$, and by noting that $\hat{v}$ is nonnegative. (b) For the upper bound, we first set $z = (r-\delta)q > 0$ and $C = \tilde{\Pi}(z)/r \geqslant 0$. Then, from (2.6), we have

$$\Pi(k) \leqslant (r-\delta)qk + rC \quad \forall k \geqslant 0.$$

This inequality implies that the function $\varphi(k) = kq + C$ satisfies $r\varphi - \mathscr{L}\varphi - \Pi \geqslant 0$, and therefore is a supersolution to (3.6). □

Lemma 4.3. (a) *The value function* $\hat{v}$ *is nondecreasing, concave, and continuous on* $(0,\infty)$.
(b) $0 \leqslant \hat{v}(0^+) \leqslant [\tilde{\Pi}((r-\delta)p)]/r$.

Proof. (a) From Lemma 4.1, we have

$$\hat{v}(k) \geqslant \hat{v}(k') + p(1-\lambda)(k-k') \quad \forall 0 \leqslant k' \leqslant k.$$

This proves in particular that $\hat{v}$ is nondecreasing.

The proof of concavity of $\hat{v}$ is standard. It is established by considering convex combinations of initial states and control and using the linearity of dynamics (2.2) and the concavity of Π. Since $\hat{v}$ is finite and concave on $(0,\infty)$, it is continuous on $(0,\infty)$.

(b) The limit $\hat{v}(0^+)$ exists from the nondecreasing property of $\hat{v}$. By taking $q=p$ in the inequality of Lemma 4.2, we obtain the required estimation on this limit. □

Since $\hat{v}$ is concave on $(0,\infty)$, it admits a right derivative $\hat{v}'_+(k)$ and a left derivative $\hat{v}'_-(x)$ at any $k>0$, and $\hat{v}'_+(k) \leqslant \hat{v}'_-(k)$. Moreover, the concavity of $\hat{v}$, combined with inequality (4.4), shows that

$$p(1-\lambda) \leqslant \hat{v}'_+(k) \leqslant \hat{v}'_-(k) \leqslant p \quad \forall k>0. \tag{4.6}$$

We then define the so-called no-transaction region

$$\mathscr{N}\mathscr{T} = \{k>0 : p(1-\lambda) < \hat{v}'_+(k) \leqslant \hat{v}'_-(k) < p\}.$$

Lemma 4.4. *There exists* $k_b \leqslant k_s \in [0, \infty]$ *such that*

$$\mathcal{N}\mathcal{T} = (k_b, k_s), \tag{4.7}$$

$\hat{v}$ *is differentiable on* $(0, k_b) \cup (k_s, \infty)$ *and*

$$\hat{v}'(k) = p, \quad \text{on } \mathcal{B} = (0, k_b) \tag{4.8}$$

$$\hat{v}'(k) = p(1-\lambda), \quad \text{on } \mathcal{S} = (k_s, \infty). \tag{4.9}$$

Proof. We set $k_b = \inf\{k \geqslant 0 : \hat{v}'_+(k) < p\}$ and $k_s = \sup\{k \geqslant k_b : v'_-(k) > p(1-\lambda)\}$. Then, $p \leqslant v'_+(k) \leqslant v'_-(k)$, for all $k < k_b$ and $v'_+(k) \leqslant v'_-(k) \leqslant p(1-\lambda)$, for all $k > k_s$. Together with (4.6), this proves (4.8) and (4.9). Finally, the concavity of $\hat{v}$ leads to (4.7). □

Remark 4.1. We will see later that $0 < k_b < k_s < \infty$, and the optimal strategy for the company consists of (i) neither increasing nor reducing capitals when it is in the region $\mathcal{N}\mathcal{T} = (k_b, k_s)$, (ii) increasing capital when it is below k_b in order to reach the threshold k_b, and (iii) reducing capital when it is above k_s, in order to attain the level k_s. The region $\mathcal{B} = (0, k_b)$ is called the *expansion region*, and $\mathcal{S} = (k_s, \infty)$, the *contraction region*.

5. Viscosity solutions and regularity of the auxiliary value function

The concept of viscosity solutions is known to be a general and powerful tool for characterizing the value function of a stochastic control problem (cf. [7]). It is based on the dynamic programming principle, which we now recall in our context.

Dynamic programming principle. For all $k > 0$, we have

$$\hat{v}(k) = \sup_{(L,M) \in \mathcal{A}_0(k)} E\left[\int_0^\theta e^{-rt}(\Pi(K_t)\,dt - p\,dL_t + (1-\lambda)p\,dM_t) + e^{-r\theta}\hat{v}(K_\theta)\right] \tag{5.1}$$

for any $\theta \in \mathcal{T}$ possibly depending on (L, M) in the supremum in (5.1). Here, we used the convention that $e^{-r\theta(\omega)} = 0$ when $\theta(\omega) = \infty$. Although this result is standard, we are not able to find a precise reference covering exactly this situation (see however [18] for a finite horizon case). Thus, we provide for the sake of completeness a quick proof in Appendix A.

Theorem 5.1. *The value function* $\hat{v}$ *is a continuous viscosity solution on* $(0, \infty)$ *of the Bellman equation* (3.6), *i.e.*, $\hat{v}$ *satisfies*:

(i) *Supersolution Viscosity Property*: *for any* $k_0 > 0$ *and any* C^2 *function* φ *in a neighborhood of* k_0 *s.t.* k_0 *is a local minimum of* $\hat{v} - \varphi$ *with* $(\hat{v} - \varphi)(k_0) = 0$,

$$\min\{r\varphi(k_0) - \mathcal{L}\varphi(k_0) - \Pi(k_0), -\varphi'(k_0) + p, \varphi'(k_0) - (1-\lambda)p\} \geqslant 0. \tag{5.2}$$

(ii) *Subsolution Viscosity Property*: *for any* $k_0 > 0$ *and any* C^2 *function* φ *in a neighborhood of* k_0 *s.t.* k_0 *is a local maximum of* $\hat{v} - \varphi$ *with* $(\hat{v} - \varphi)(k_0) = 0$,

$$\min\{r\varphi(k_0) - \mathscr{L}\varphi(k_0) - \Pi(k_0), -\varphi'(k_0) + p, \varphi'(k_0) - (1-\lambda)p\} \leqslant 0. \tag{5.3}$$

Proof. The proof is based on the dynamic programming principle and Itô's formula. It is standard, but somewhat technical in this singular control context, and is deferred to the Appendix B. □

Based on the property that the value function is a concave viscosity solution of the Bellman equation, we can now prove that it is in fact C^2.

Theorem 5.2. *The value function* $\hat{v}$ *is a classical* C^2 *solution on* $(0,\infty)$ *to the Bellman equation*

$$\min\{r\hat{v} - \mathscr{L}\hat{v} - \Pi(k), -\hat{v}'(k) + p, \hat{v}'(k) - (1-\lambda)p\} = 0, \quad k > 0.$$

Proof. *Step* 1: We first prove that $\hat{v}$ is a C^1 function on $(0,\infty)$. Since $\hat{v}$ is concave, the left and right derivatives $\hat{v}'_-(k)$ and $\hat{v}'_+(k)$ exist for any $k > 0$ and $\hat{v}'_+(k) \leqslant \hat{v}'_-(k)$. We proceed by contradiction argument. Suppose that $\hat{v}'_+(k_0) < \hat{v}'_-(k_0)$ for some $k_0 > 0$. Fix some q in $(\hat{v}'_+(k_0), \hat{v}'_-(k_0))$ and consider the function

$$\varphi_\varepsilon(k) = \hat{v}(k_0) + q(k - k_0) - \frac{1}{2\varepsilon}(k - k_0)^2$$

with $\varepsilon > 0$. Then k_0 is a local maximum of $(\hat{v} - \varphi_\varepsilon)$ with $\varphi_\varepsilon(k_0) = \hat{v}(k_0)$. Since $\varphi'_\varepsilon(k_0) = q \in (p(1-\lambda), p)$ and $\varphi''_\varepsilon(k_0) = 1/\varepsilon$, the subsolution inequality (5.3) implies that we must have

$$r\varphi(k_0) - \delta k_0 q + \frac{1}{\varepsilon} - \Pi(k_0) \leqslant 0. \tag{5.4}$$

Choosing ε sufficiently small leads to a contradiction. Hence, $\hat{v}'_+(k_0) = \hat{v}'_-(k_0)$ for all $k_0 > 0$.

Step 2: From Lemma 4.4, $\hat{v}$ is C^2 on $(0, k_b) \cup (k_s, \infty)$ and satisfies $\hat{v}'(k) = p$, $k \in (0, k_b)$, $\hat{v}'(k) = kp(1-\lambda)$, $k \in (k_s, \infty)$. From Step 1, we have $\mathscr{N}\mathscr{T} = (k_b, k_s) = \{k > 0 : p(1-\lambda) < \hat{v}'(k) < p\}$. We now check that $\hat{v}$ is a viscosity solution of

$$r\hat{v} - \mathscr{L}\hat{v} - \Pi = 0 \quad \text{on } (k_b, k_s). \tag{5.5}$$

Let $k_0 \in (k_b, k_s)$ and φ be a C^2 function on (k_b, k_s) such that k_0 is a local maximum of $\hat{v} - \varphi$, with $(\hat{v} - \varphi)(k_0) = 0$. Since $\varphi'(k_0) = \hat{v}'(k_0) \in (p(1-\lambda), p)$, the subsolution inequality (5.3) implies that

$$r\varphi(k_0) - \mathscr{L}\varphi(k_0) - \Pi(k_0) \leqslant 0.$$

This proves that $\hat{v}$ is a viscosity supersolution of (5.5) on (k_b, k_s). The proof of the supersolution viscosity property is similar. Now for arbitrary $k_1 \leqslant k_2 \in (k_b, k_s)$, consider the Dirichlet boundary problem:

$$rV - \mathscr{L}V - \Pi(k) = 0 \quad \text{on } (k_1, k_2), \tag{5.6}$$

$$V(k_1) = \hat{v}(k_1), \quad V(k_2) = \hat{v}(k_2). \tag{5.7}$$

Classical results provide the existence and uniqueness of a C^2 function V on (k_1, k_2) as a solution to (5.6) and (5.7). In particular, this smooth function V is a viscosity solution of (5.5) on (k_1, k_2). From standard uniqueness results on viscosity solutions (here, for a linear PDE in a bounded domain), we deduce that $\hat{v} = V$ on (k_1, k_2). From the arbitrariness of k_1 and k_2, it is clear that $\hat{v}$ is C^2 on (k_b, k_s) and satisfies (5.5) in the classical sense.

Step 3: It remains to prove the C^2 property at k_b and k_s in the case $0 < k_b < k_s < \infty$. Let $k \in (0, k_b)$. Since $\hat{v}$ is C^2 on $(0, k_b)$ with $\hat{v}'(k) = p$, the supersolution inequality (5.2) applied to the point k and the test function $\varphi = \hat{v}$ implies that v satisfies the following in the classical sense:

$$r\hat{v}(k) - \mathscr{L}\hat{v}(k) - \Pi(k) \geqslant 0, \quad 0 < k < k_b.$$

Since the derivative of $\hat{v}$ is the constant p on $(0, k_b)$, this yields

$$r\hat{v}(k) - \delta k p - \Pi(k) \geqslant 0, \quad 0 < k < k_b$$

and

$$r\hat{v}(k_b) - \delta k_b p - \Pi(k_b) \geqslant 0. \tag{5.8}$$

On the other hand, from the C^1 smooth-fit at k_b, we have, by sending k downwards k_b in (5.5)

$$r\hat{v}(k_b) - \delta k_b p - \Pi(k_b) = \tfrac{1}{2}\gamma^2 k_b \hat{v}''(k_b^+). \tag{5.9}$$

From the concavity of $\hat{v}$, the RHS of (5.9) is nonpositive, which combined with (5.8), implies that $\hat{v}''(k_b^+) = 0$. This proves that $\hat{v}$ is C^2 at k_b with

$$\hat{v}''(k_b) = 0.$$

Similar arguments prove the C^2 condition of $\hat{v}$ at k_s with

$$\hat{v}''(k_s) = 0. \qquad \square$$

6. Solution to the auxiliary optimization problem

6.1. Some preliminary results on an ODE

We recall some useful results on the second-order linear differential equation

$$r\hat{v} - \mathscr{L}\hat{v} - \Pi = 0, \tag{6.1}$$

arising in the Bellman equation (3.6).

It is well known that the general solution to the ODE (6.1) with $\Pi = 0$ is given by

$$\hat{V}(k) = Ak^m + Bk^n,$$

where under A3: $r > \delta$

$$m = \frac{-\delta}{\gamma^2} + \frac{1}{2} - \sqrt{\left(\frac{-\delta}{\gamma^2} + \frac{1}{2}\right)^2 + \frac{2r}{\gamma^2}} < 0,$$

$$n = \frac{-\delta}{\gamma^2} + \frac{1}{2} + \sqrt{\left(\frac{-\delta}{\gamma^2} + \frac{1}{2}\right)^2 + \frac{2r}{\gamma^2}} > 1$$

are the roots of

$$\tfrac{1}{2}\gamma^2 m(m-1) + \delta m - r = 0.$$

Moreover, the ODE (6.1) admits a twice continuously differentiable particular solution on $(0,\infty)$ given by (cf. [9])

$$\hat{V}_0(k) = E\left[\int_0^\infty \mathrm{e}^{-rt}\Pi(\hat{K}_t)\,\mathrm{d}t\right] = k^n G_1(k) + k^m G_2(k)$$

with

$$G_1(k) = \frac{2}{\gamma^2(n-m)}\int_k^\infty s^{-n-1}\Pi(s)\,\mathrm{d}s, \quad k>0, \tag{6.2}$$

$$G_2(k) = \frac{2}{\gamma^2(n-m)}\int_0^k s^{-m-1}\Pi(s)\,\mathrm{d}s, \quad k>0. \tag{6.3}$$

Note that the integrals defining G_1 and G_2 are well-defined and finite for $k>0$ by the linear growth condition (2.6) on Π.

Under assumptions A1 and A2, the limiting behavior of $\hat{V}_0$ when k goes to zero is stated as follows.

Lemma 6.1. $\hat{V}_0(0^+) = 0$ *and* $\hat{V}_0'(0^+) = \infty$.

Proof. (i) From (2.6), we have for all $s,k>0$,

$$\Pi(s) \leqslant \tilde{\Pi}\left(\frac{1}{\sqrt{k}}\right) + \frac{s}{\sqrt{k}}.$$

We then get for all $k>0$,

$$\int_k^\infty s^{-n-1}\Pi(s)\,\mathrm{d}s \leqslant \tilde{\Pi}\left(\frac{1}{\sqrt{k}}\right)\frac{1}{nk^n} + \frac{1}{\sqrt{k}}\,\frac{1}{(n-1)k^{n-1}}.$$

It follows that

$$k^n G_1(k) \leqslant \frac{2}{\gamma^2(n-m)}\left[\frac{1}{n}\tilde{\Pi}\left(\frac{1}{\sqrt{k}}\right) + \frac{\sqrt{k}}{n-1}\right].$$

Recalling the well-known duality relation $\tilde{\Pi}(\infty) = \Pi(0)$ (cf. [15]), and since $\Pi(0) = 0$, we deduce by sending k to zero in the last inequality that

$$\lim_{k\downarrow 0} k^n G_1(k) = 0.$$

On the other hand, using the nondecreasing property of Π, it is easy to see that

$$k^m G_2(k) \leqslant \frac{2}{\gamma^2(n-m)}\,\frac{-\Pi(k)}{m},$$

so that

$$\lim_{k\downarrow 0} k^m G_2(k) = 0.$$

This proves $\lim_{k\downarrow 0} \hat{V}_0(k) = 0$.

(ii) In view of point (i), we have to prove that $\lim_{k\downarrow 0} \hat{V}_0(k)/k = \infty$. From the standing assumption (2.5), for any arbitrary large $N > 0$, there exists $\bar{k} > 0$, such that for all $0 < k < \bar{k}$, $\Pi(k)/k \geqslant N$. Hence, we deduce that for all $0 < k < \bar{k}$

$$\frac{\hat{V}_0(k)}{k} \geqslant k^{m-1} G_2(k) \geqslant \frac{2}{\gamma^2(n-m)} \frac{N}{1-m}.$$

This proves the required result. □

6.2. Construction of the auxiliary value function

Lemma 6.2. *The expansion and contraction thresholds satisfy*

$$0 < k_b < k_s < \infty.$$

Proof. We first check that $k_s > 0$. Otherwise, the expansion and no-transaction regions are empty and $\hat{v}$ is of the form

$$\hat{v}(k) = kp(1-\lambda) + \hat{v}(0^+) \quad \forall k > 0.$$

Since $\hat{v}$ satisfies the Bellman equation (3.6), we must have $r\hat{v} - \mathscr{L}\hat{v} - \Pi \geqslant 0$, which implies

$$r\hat{v}(0^+) \geqslant \sup_{k>0}(\Pi(k) - (r-\delta)kp(1-\lambda)) = \tilde{\Pi}((r-\delta)p(1-\lambda)).$$

Since $\tilde{\Pi}$ is nonincreasing, this contradicts the upper bound for $\hat{v}(0^+)$ in Lemma 4.3.

We now check that $k_b > 0$. If not, the expansion region would be empty, and we would have from Lemma 4.4 and Theorem 5.2

$$r\hat{v} - \mathscr{L}\hat{v} - \Pi = 0, \quad 0 < k < k_s.$$

Hence, $\hat{v}$ would satisfy on the nonempty set $(0, k_s)$

$$\hat{v}(k) = Ak^m + Bk^n + \hat{V}_0(k), \quad 0 < k < k_s.$$

Since $m < 0$ and $|\hat{v}(0^+)| < \infty$, this implies $A = 0$. Now, since $n > 1$, we get $\hat{v}'(0^+) = \hat{V}_0'(0^+) = \infty$, a contradiction to the fact that $\hat{v}'(k) \leqslant p$ for all $k > 0$.

We also have $k_b < \infty$. Otherwise, $\hat{v}$ would be of the form

$$\hat{v}(k) = kp + \hat{v}(0^+) \quad \forall k > 0.$$

This contradicts the growth condition (4.5).

We claim that $k_s < \infty$. On the contrary, the contraction region would be empty, and we would have from Lemma 4.4 and Theorem 5.2

$$r\hat{v} - \delta k\hat{v}'(k) - \tfrac{1}{2}\gamma^2 k^2 \hat{v}''(k) - \Pi(k) = 0 \quad \forall k > k_b.$$

By the concavity of $\hat{v}$, this implies:

$$\delta \hat{v}'(k) \geqslant r\,\frac{\hat{v}(k)}{k} - \frac{\Pi(k)}{k} \quad \forall k > k_b. \tag{6.4}$$

We also get that $q := \lim_{k\to\infty} \hat{v}'(k)$ exists and lies in $[p(1-\lambda), p]$, so that by the L'Hopital rule, $\hat{v}(k)/k$ goes to q as k goes to infinity. Moreover, from (2.6), we know that $\Pi(k)/k$ goes to 0 as k tends to infinity. Hence, by sending k to infinity in (6.4), we obtain $\delta q \geqslant rq$, a contradiction to A3.

Finally, since $\hat{v}$ is C^1, we have $\hat{v}'(k_b) = p$ and $\hat{v}'(k_s) = p(1-\lambda)$, which implies $k_b < k_s$. $\square$

We can now explicitly determine the auxiliary value function $\hat{v}$.

Theorem 6.1. *The value function $\hat{v}$ has the following structure*:

$$\hat{v}(k) = \begin{cases} kp + \hat{v}(0^+), & k \leqslant k_b, \\ Ak^m + Bk^n + \hat{V}_0(k), & k_b < k < k_s, \\ kp(1-\lambda) + C_1, & k \geqslant k_s, \end{cases} \tag{6.5}$$

where the six-tuple of constants $(A, B, k_b, k_s, \hat{v}(0^+), C_1)$ is the unique solution to the system of equations:

$$Ak_b^m + Bk_b^n + \hat{V}_0(k_b) = k_b p + \hat{v}(0^+), \tag{6.6}$$

$$Ak_s^m + Bk_s^n + \hat{V}_0(k_s) = k_s p(1-\lambda) + C_1, \tag{6.7}$$

$$mAk_b^{m-1} + nBk_b^{n-1} + \hat{V}_0'(k_b) = p, \tag{6.8}$$

$$mAk_s^{m-1} + nBk_s^{n-1} + \hat{V}_0'(k_s) = p(1-\lambda), \tag{6.9}$$

$$m(m-1)Ak_b^{m-2} + n(n-1)Bk_b^{n-2} + \hat{V}_0''(k_b) = 0, \tag{6.10}$$

$$m(m-1)Ak_s^{m-2} + n(n-1)Bk_s^{n-2} + \hat{V}_0''(k_s) = 0, \tag{6.11}$$

resulting from the continuity and smooth-fit C^1 and C^2 conditions of $\hat{v}$ at k_b and k_s.

Proof. We know from Lemma 4.4 that on $(0, k_b)$ and (k_s, ∞) (which are nonempty sets by Lemma 6.2), $\hat{v}$ has the structure described in (6.5). Moreover, on (k_b, k_s), we have $p(1-\lambda) < \hat{v}' < p$ from Lemma 4.4. Therefore, by Theorem 5.2, $\hat{v}$ satisfies $r\hat{v} - \mathscr{L}\hat{v} - \Pi = 0$, and from the previous Theorem 6.1, has the form written in (6.5). We know the existence of a six-tuple $(A, B, k_b, k_s, \hat{v}(0^+), C_1)$ solution to the system of equations (6.6)–(6.11). Indeed, this results from the continuity and smooth-fit C^1 and C^2 conditions of $\hat{v}$ at k_b and k_s that we proved to hold true. On the other hand, suppose that one can find another six-tuple $(\tilde{A}, \tilde{B}, \tilde{k}_b, \tilde{k}_s, \tilde{v}(0^+), \tilde{C}_1)$ solution to (6.6)–(6.11). Then, by considering the function

$$\tilde{v}(k) = \begin{cases} kp + \tilde{v}(0^+), & k \leqslant \tilde{k}_b, \\ \tilde{A}k^m + \tilde{B}k^n + \hat{V}_0(k), & \tilde{k}_b < k < \tilde{k}_s, \\ kp(1-\lambda) + \tilde{C}_1, & k \geqslant \tilde{k}_s, \end{cases}$$

it turns out that $\tilde{v}$ is a C^2 solution (hence also a viscosity solution) on $(0, \infty)$, with linear growth condition, to the same HJB equation (3.6) than $\hat{v}$, i.e.

$$\min\{r\tilde{v} - \mathscr{L}\tilde{v} - \Pi, -\tilde{v}' + p, \tilde{v}' - (1-\lambda)p\} = 0, \quad \text{on} \ \ (0, \infty). \tag{6.12}$$

From standard uniqueness results for the PDE (3.6) (see e.g. [8]), we deduce that $\tilde{v} = v$. This proves that $(\tilde{A}, \tilde{B}, \tilde{k}_b, \tilde{k}_s, \tilde{v}(0^+), \tilde{C}_1) = (A, B, k_b, k_s, \hat{v}(0^+), C_1)$. □

Remark 6.1. The value function $\hat{v}$ satisfies in (k_b, k_s):

$$r\hat{v}(k) - \delta k\hat{v}'(k) - \tfrac{1}{2}\gamma^2 k^2 \hat{v}''(k) - \Pi(k) = 0, \quad k \in (k_b, k_s).$$

From the continuity and smooth C^1 and C^2 conditions of $\hat{v}$ at k_b, i.e., $\hat{v}(k_b) = k_b p + \hat{v}(0^+)$, $\hat{v}'(k_b) = p$ and $\hat{v}''(k_b) = 0$, we then deduce

$$(r-\delta)k_b p + r\hat{v}(0^+) = \Pi(k_b). \tag{6.13}$$

Similarly, the threshold k_s satisfies

$$(r-\delta)k_s p(1-\lambda) + rC_1 = \Pi(k_s). \tag{6.14}$$

Remark 6.2. *Computation of* $\hat{v}$: From a computational viewpoint, the constants $A, B, k_b, k_s, \hat{v}(0^+), C_1$ may be determined as follows. From Eqs. (6.8) and (6.10), we express A and B in terms of k_b

$$mA = \frac{k_b^{-m}}{n-m}(p(n-1)k_b - (n-1)k_b\hat{V}_0'(k_b) + k_b^2\hat{V}_0''(k_b)), \tag{6.15}$$

$$nB = \frac{k_b^{-n}}{n-m}(p(1-m)k_b - (1-m)k_b\hat{V}_0'(k_b) - k_b^2\hat{V}_0''(k_b)). \tag{6.16}$$

We proceed similarly for Eqs. (6.9) and (6.11) and get

$$mA = \frac{k_s^{-m}}{n-m}(p(1-\lambda)(n-1)k_s - (n-1)k_s\hat{V}_0'(k_s) + k_s^2\hat{V}_0''(k_s)), \tag{6.17}$$

$$nB = \frac{k_b^{-n}}{n-m}(p(1-\lambda)(1-m)k_s - (1-m)k_s\hat{V}_0'(k_s) - k_s^2\hat{V}_0''(k_s)). \tag{6.18}$$

Comparing (6.15)–(6.17) and (6.16)–(6.18) yields that k_b and k_s are solutions to

$$\begin{aligned} &p(n-1)k_b^{1-m} - (n-1)k_b^{1-m}\hat{V}_0'(k_b) + k_b^{2-m}\hat{V}_0''(k_b) \\ &\quad = p(1-\lambda)(n-1)k_s^{1-m} - (n-1)k_s^{1-m}\hat{V}_0'(k_s) + k_s^{2-m}\hat{V}_0''(k_s), \end{aligned} \tag{6.19}$$

$$p(1-m)k_b^{1-n} - (1-m)k_b^{1-n}\hat{V}_0'(k_b) - k_b^{2-n}\hat{V}_0''(k_b)$$
$$= p(1-\lambda)(1-m)k_s^{1-n} - (1-m)k_s^{1-n}\hat{V}_0'(k_s) - k_s^{2-n}\hat{V}_0''(k_s). \quad (6.20)$$

6.3. Special case of power profit function

We consider the case where Π is a Cobb–Douglas profit function. Without loss of generality, we assume that $\Pi(k) = k^\alpha$, with $0<\alpha< 1$. Then

$$\hat{V}_0(k) = Dk^\alpha \quad \text{with} \quad D = \frac{1}{r - \alpha\delta + (\gamma^2/2)\alpha(1-\alpha)}.$$

Two Eqs. (6.19) and (6.20) determining k_b and k_s are written as

$$p(n-1)k_b^{1-m} - (n-\alpha)D\alpha k_b^{\alpha-m} = p(1-\lambda)(n-1)k_s^{1-m} - (n-\alpha)D\alpha k_s^{\alpha-m}, \quad (6.21)$$

$$p(1-m)k_b^{1-n} - (\alpha-m)D\alpha k_b^{\alpha-n} = p(1-\lambda)(1-m)k_s^{1-n} - (\alpha-m)D\alpha k_s^{\alpha-n}. \quad (6.22)$$

By denoting $y = k_s/k_b$ (>1), Eqs. (6.21) and (6.22) are written equivalently in terms of y and k_b as

$$p(n-1)[y^m - (1-\lambda)y] = (n-\alpha)D\alpha k_b^{\alpha-1}[y^m - y^\alpha],$$

$$p(1-m)[y^n - (1-\lambda)y] = (\alpha-m)D\alpha k_b^{\alpha-1}[y^n - y^\alpha].$$

By comparing both expressions of $D\alpha k_b^\alpha$ in these two previous equations, we obtain an equation for y:

$$F(y) := \frac{(n-\alpha)(1-m)y^{n-1}(y^\alpha - y^m) + (\alpha-m)(n-1)y^{m-1}(y^n - y^\alpha)}{(n-\alpha)(1-m)(y^\alpha - y^m) + (\alpha-m)(n-1)(y^n - y^\alpha)} = 1-\lambda. \quad (6.23)$$

It is easy to see that

$$\lim_{y\downarrow 1} F(y) = 1 \quad \text{and} \quad \lim_{y\to\infty} F(y) = 0. \quad (6.24)$$

By the continuity of $F(y)$, we get that the existence of $y_\lambda > 1$ s.t. Eq. (6.23) holds, i.e. $F(y_\lambda) = 1-\lambda$. The uniqueness of such y_λ follows from Remark 6.2.

We can then express all the parameters A, B, k_b, k_s, $\hat{v}(0^+)$ and C_1 in terms of y_λ:

$$k_b = \left(\frac{p(1-m)(y_\lambda^n - (1-\lambda)y_\lambda)}{\alpha D(\alpha-m)(y_\lambda^n - y_\lambda^\alpha)}\right)^{1/(\alpha-1)}, \quad k_s = k_b y_\lambda,$$

$$A = \frac{(n-1)pk_b - \alpha D(\alpha-n)k_b^\alpha}{m(n-m)k_b^m}, \quad B = \frac{(1-m)pk_b - \alpha D(\alpha-m)k_b^\alpha}{n(n-m)k_b^n},$$

$$\hat{v}(0^+) = Ak_b^m + Bk_b^n + Dk_b^\alpha - pk_b, \quad C_1 = Ak_s^m + Bk_b^n + Dk_s^\alpha - pk_s(1-\lambda).$$

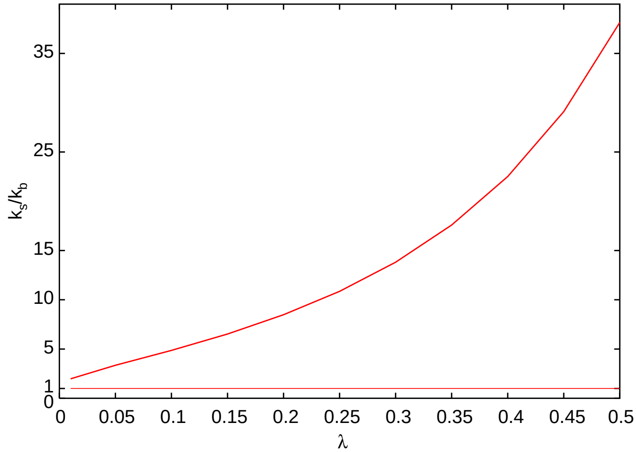

Fig. 1. Relation of k_s/k_b vs. λ.

Remark 6.3. From the above expression of k_b, we see that k_b is decreasing with respect to p and goes to ∞ when p goes to 0. We also see numerically that the ratio k_s/k_b increases with λ. Moreover, from (6.24), k_s/k_b goes to 1 when λ goes to 0, and goes to infinity when λ goes to 1. We discuss the corresponding economic interpretation later in Section 8.

A numerical example. Take $\delta = 1,\ \gamma = 2,\ r = 3,\ \alpha = 0.6$. Fig. 1 shows the graph of k_s/k_b as an increasing function of λ.

On the other hand, for the values $\lambda = 0.6$ and $p = 0.5$, we obtain $k_b = 0.069$, $k_s = 4.8$, and the value function $\hat{v}$:

$$\hat{v}(k) = \begin{cases} 0.5k + 0.044, & k \leqslant 0.069, \\ 0.00005k^{-1} + 0.027k^{1.5} + 0.347k^{0.6}, & 0.069 < k \leqslant 4.8, \\ 0.2k + 0.214, & k > 4.8. \end{cases} \tag{6.25}$$

7. Solution to the original problem and construction of optimal policies and entry decision

7.1. Solution to the optimal stopping problem

We now return to the original value function w given by the optimal stopping problem (3.5). We first state some elementary properties of w.

Lemma 7.1. (a) *The value function w is nondecreasing on* $(0, \infty)$.

(b) *w is p-Lipschitz continuous on* $(0, \infty)$:

$$|w(k) - w(k')| \leqslant p|k - k'| \quad \forall k, k' > 0.$$

(c) *w satisfies*

$$(\hat{v}(0^+) - C_I)^+ \leqslant w(k) \leqslant (\hat{v}(0^+) - C_I)^+ + pk, \quad k > 0. \tag{7.1}$$

In particular, we have $w(0^+) = (\hat{v}(0^+) - C_I)^+ = \max(0, \hat{v}(0^+) - C_I)$.

Proof. (a) That v is nondecreasing follows from the corresponding properties of $\hat{v}$, together with the fact that the solution to (3.4) is written explicitly as $\hat{K}_t = ke^{\delta t}N_t$, where N is the exponential martingale given by

$$N_t = \exp\left(-\frac{\gamma^2}{2}t + \gamma W_t\right).$$

(b) From the p-Lipschitz continuity of $\hat{v}$, we have

$$\begin{aligned}|w(k) - w(k')| &\leqslant \sup_{\tau_I \in \mathcal{T}} E[e^{-r\tau_I}|\hat{v}(ke^{\delta\tau_I}N_{\tau_I}) - \hat{v}(k'e^{\delta\tau_I}N_{\tau_I})|] \\ &\leqslant p|k - k'| \sup_{\tau_I \in \mathcal{T}} EN_{\tau_I},\end{aligned}$$

since $r - \delta > 0$. We get the required result by noting that $\sup_{\tau_I \in \mathcal{T}} EN_{\tau_I} = 1$.

(c) Since $\hat{v}(0^+) \leqslant \hat{v}(k) \leqslant \hat{v}(0^+) + pk$, for all $k > 0$, we have

$$\sup_{\tau_I \in \mathcal{T}} E[e^{-r\tau_I}(\hat{v}(0^+) - C_I)] \leqslant w(k) \leqslant \sup_{\tau_I \in \mathcal{T}} E[e^{-r\tau_I}(\hat{v}(0^+) - C_I)] + p \sup_{\tau_I \in \mathcal{T}} E[e^{-r\tau_I}\hat{K}_{\tau_I}].$$

We conclude by noting that

$$\sup_{\tau_I \in \mathcal{T}} E[e^{-r\tau_I}(\hat{v}(0^+) - C_I)] = (\hat{v}(0^+) - C_I)_+,$$

$$\sup_{\tau_I \in \mathcal{T}} E[e^{-r\tau_I}\hat{K}_{\tau_I}] \leqslant k \sup_{\tau_I \in \mathcal{T}} EN_{\tau_I} = k. \qquad \square$$

From classical results on optimal stopping theory, we have the following characterization of w.

Theorem 7.1. *w is a continuous viscosity solution of*

$$\min\{rw - \mathcal{L}w, w + C_I - \hat{v}\} = 0. \tag{7.2}$$

We introduce the entry region

$$\mathcal{E} = \{k > 0 : w(k) = \hat{v}(k) - C_I\}$$

and the infimum of this set

$$k_I = \inf \mathcal{E}.$$

$\mathcal{E}$ is a closed subset in $(0, \infty)$ and corresponds to the region where it is optimal for the company to activate its production. It is well-known that in the continuation region, i.e. the complement set $\mathcal{E}^c$ of $\mathcal{E}$ in $(0, \infty)$,

$$\mathcal{E}^c = \{k > 0 : w(k) > \hat{v}(k) - C_I\},$$

the value function w is smooth C^2 and satisfies

$$rw(k) - \mathcal{L}w(k) = 0 \quad \forall k \in \mathcal{E}^c. \tag{7.3}$$

However, w does not inherit the concavity property of $\hat{v}$. Therefore, the regularity property of w and connected property of $\mathscr{E}$ cannot be proved directly from classical results. We will carefully analyze the entry region and the value function w.

We first recall a well-known useful result on the entry region.

Lemma 7.2.

$$\mathscr{E} \subset Q = \{k > 0 : r\hat{v} - \mathscr{L}\hat{v} - rC_I \geqslant 0\}.$$

Proof. For the sake of completeness, we outline the key idea for the proof. Let $k_0 \in \mathscr{E}$. Then, k_0 is a minimum of $w - \varphi$ with $w(k_0) = \varphi(k_0)$, where $\varphi = \hat{v} - C_I$ is C^2. By the supersolution viscosity property of w, we deduce that $r\varphi(k_0) - \mathscr{L}\varphi(k_0) \geqslant 0$, i.e., $k_0 \in Q$. □

Lemma 7.3. *We have the following characterization of* Q:

- *If* $rC_I \leqslant \Pi(k_b)$, *then*

$$Q = \left[\frac{r}{r-\delta}\frac{C_I - \hat{v}(0^+)}{p}, \infty\right) \cap (0, \infty). \tag{7.4}$$

- *If* $\Pi(k_b) < rC_I < \Pi(k_s)$, *then*

$$Q = [\Pi^{-1}(rC_I), \infty). \tag{7.5}$$

- *If* $rC_I \geqslant \Pi(k_s)$, *then*

$$Q = \left[\frac{r}{r-\delta}\frac{C_I - C_1}{p(1-\lambda)}, \infty\right). \tag{7.6}$$

Proof. We partition the subset Q into three regions $Q = Q_1 \cup Q_2 \cup Q_3$ where

$$\begin{aligned} Q_1 &= \{0 < k \leqslant k_b : r\hat{v} - \mathscr{L}\hat{v} - rC_I \geqslant 0\} \\ &= \{0 < k \leqslant k_b : rC_I \leqslant (r-\delta)kp + \hat{v}(0^+)\}, \end{aligned}$$

since $\hat{v}(k) = kp + \hat{v}(0^+)$ on $(0, k_b)$,

$$\begin{aligned} Q_2 &= \{k_b < k < k_s : r\hat{v} - \mathscr{L}\hat{v} - rC_I \geqslant 0\} \\ &= \{k_b < k < k_s : rC_I \leqslant \Pi(k)\}, \end{aligned}$$

since $\hat{v}$ satisfies $r\hat{v} - \mathscr{L}\hat{v} - \Pi = 0$ on (k_b, k_s), and

$$\begin{aligned} Q_3 &= \{k > k_s : r\hat{v} - \mathscr{L}\hat{v} - rC_I \geqslant 0\} \\ &= \{k > k_s : rC_I \leqslant (r-\delta)kp(1-\lambda) + rC_1\}, \end{aligned}$$

since $\hat{v}(k) = kp(1-\lambda) + C_1$ on (k_s, ∞).

We now distinguish the following three cases.

Case 1: $rC_I \leqslant \Pi(k_b)$. Then from expression (6.13) of k_b, we have $rC_I \leqslant (r-\delta)k_b p + \hat{v}(0^+)$, and thus

$$Q_1 = \left[\frac{r}{r-\delta}\frac{C_I - \hat{v}(0^+)}{p}, k_b\right] \cap (0,\infty).$$

Moreover, since Π is nondecreasing, we have $Q_2 = (k_b, k_s)$. Finally, from expression (6.14) of k_s, we have for all $k \geqslant k_s$, $(r-\delta)kp(1-\lambda) + rC_1 \geqslant \Pi(k_s) \geqslant \Pi(k_b) \geqslant rC_I$, and hence $Q_3 = (k_s, \infty)$. This proves equality (7.4).

Case 2: $\Pi(k_b) < rC_I < \Pi(k_s)$. Then from expression (6.13) of k_b, $Q_1 = \emptyset$. We also have that $Q_2 = [\Pi^{-1}(rC_I), k_s)$ since Π is nondecreasing. Finally, by the same argument as in the previous case, from expression (6.14) of k_s, we have $Q_3 = (k_s, \infty)$. This proves equality (7.5).

Case 3: $rC_I \geqslant \Pi(k_s)$. Then, Q_1 and Q_2 are empty and again using expression (6.14) of k_s, we have

$$Q_3 = \left[\frac{r}{r-\delta}\frac{C_I - C_1}{p(1-\lambda)}, \infty\right).$$

This completes the proof. □

Remark 7.1. When $C_I \leqslant \hat{v}(0^+)$, we have $rC_I \leqslant \Pi(k_b)$ by (6.13). Therefore, by Lemma 7.3, $Q = (0,\infty)$. When $C_I > \hat{v}(0^+)$, the previous Lemma shows that $Q = [\bar{\kappa}_I, \infty)$, with $\bar{\kappa}_I > 0$ given by

$$\bar{\kappa}_I = \begin{cases} \dfrac{r}{r-\delta}\dfrac{C_I - \hat{v}(0^+)}{p} & \text{if } \ rC_I \leqslant \Pi(k_b), \\ \Pi^{-1}(rC_I) & \text{if } \Pi(k_b) \leqslant rC_I \leqslant \Pi(k_s), \\ \dfrac{r}{r-\delta}\dfrac{C_I - C_1}{p(1-\lambda)} & \text{if } \ rC_I \geqslant \Pi(k_s). \end{cases}$$

We can now provide an explicit solution to the optimal stopping problem (3.5).

Theorem 7.2. (1) *If* $C_I \leqslant \hat{v}(0^+)$, *then* $k_I = 0$ *and* $\mathscr{E} = (0,\infty)$, *i.e.*, $w = \hat{v} - C_I$.

(2) *If* $C_I > \hat{v}(0^+)$, *then* $k_I \in (0,\infty)$ *and* $\mathscr{E} = [k_I, \infty)$. *The value function* w *is continuously differentiable at* k_I *and is given by*

$$w(k) = \begin{cases} (\hat{v}(k_I) - C_I)\left(\dfrac{k}{k_I}\right)^n, & k < k_I, \\ \hat{v}(k) - C_I, & k \geqslant k_I. \end{cases}$$

The entry-threshold k_I *is the unique solution in* $(0,\infty)$ *to* $G(k_I) = C_I$ *where*

$$G(k) = \hat{v}(k) - \frac{k}{n}\hat{v}'(k). \tag{7.7}$$

(3) *In both cases, the optimal entry decision is given by* $\tau_I^* = \inf\{t \geqslant 0 : \hat{K}_t \geqslant k_I\}$.

Proof. (1) *Case* $C_I \leqslant \hat{v}(0^+)$. Then, we have seen in Remark 7.1 that $Q = (0, \infty)$. By definition of Q, this implies that the C^2 function $\tilde{w} = \hat{v} - C_I$ is solution to (7.2) on $(0, \infty)$. Standard verification result then shows $\tilde{w} = w$. This means $k_I = 0$, $\mathscr{E} = (0, \infty)$ and $w = \hat{v} - C_I$.

(2) *Case* $C_I > \hat{v}(0^+)$.

Step 1: From Lemma 7.1(c), $w(0^+) = 0 > (\hat{v} - C_I)(0^+)$. By the continuity of functions w and $\hat{v} - C_I$, we have $k_I > 0$. We claim that $k_I < \infty$. Otherwise, we should have $\mathscr{E}^c = (0, \infty)$, and so the function w is smooth on $(0, \infty)$ and satisfies

$$rw - \mathscr{L}w = 0, \quad \text{on } (0, \infty).$$

Hence, w would be of the form $w(k) = Ak^m + Bk^n$, $k > 0$. Since $m < 0$ and $w(0^+) = 0$, A is equal to zero, and so $w(k) = Bk^n$. Since $n > 1$, this contradicts the linear growth condition of w stated in Lemma 7.1(c). From the continuity of w and $\hat{v}$, we deduce that $k_I \in \mathscr{E}$ and lies in Q, from Lemma 7.2. Moreover, we know from Remark 7.3 that $Q = [\bar{\kappa}_I, \infty)$. Therefore, we get $k_I \geqslant \bar{\kappa}_I$ and $[k_I, \infty) \subset Q$, which means that

$$r\hat{v}(k) - \mathscr{L}\hat{v}(k) - rC_I \geqslant 0 \quad \forall k \geqslant k_I. \tag{7.8}$$

Step 2: We now prove that w is C^1 at k_I. Recall that w is smooth C^2 on $\mathscr{E}^c$ and in the interior of $\mathscr{E}$. Hence, it admits a left and right derivative w'_- and w'_+ at k_I. Moreover, since k_I is a minimum of $w - (\hat{v} - C_I)$ with $w(k_I) = \hat{v}(k_I) - C_I$, we have $w'_-(k_I) \leqslant \hat{v}'(k_I) \leqslant w'_+(k_I)$. So, if we suppose that w were not C^1 at k_I, we could find some $q \in (w'_-(k_I), w'_+(k_I))$. Then consider the function

$$\varphi_\varepsilon(k) = w(k_I) + q(k - k_I) + \frac{1}{2\varepsilon}(k - k_I)^2$$

with $\varepsilon > 0$. Here k_I is a local minimum of $w - \varphi_\varepsilon$, with $\varphi_\varepsilon(k_I) = w(k_I) = \hat{v}(k_I) - C_I$. Hence, the supersolution viscosity property of w for the Bellman equation (7.2) implies that

$$r\varphi_\varepsilon(k_I) - \delta k_I q - \frac{1}{\varepsilon} \geqslant 0.$$

Choosing ε as sufficiently small leads to a contradiction. So w is C^1 at k_I with $w'(k_I) = \hat{v}'(k_I)$.

Step 3: For $k < k_I$, we recall that w satisfies $rw - \mathscr{L}w = 0$, and $w(0^+) = 0$. Hence, there exists a constant B s.t. $w(k) = Bk^n$, $0 < k < k_I$. The continuity and the smooth-fit C^1 condition on w at k_I imply that k_I satisfies (7.7) and $B = (\hat{v}(k_I) - C_I)/k_I^n$. So, on $(0, k_I)$, we have $w(k) = (\hat{v}(k_I) - C_I)(k/k_I)^n$. We now prove that on $[k_I, \infty)$, we have $w = \hat{v} - C_I$, i.e., $\mathscr{E} = [k_I, \infty)$. To this end, let us consider the function

$$\tilde{w}(k) = \begin{cases} (\hat{v}(k_I) - C_I)\left(\dfrac{k}{k_I}\right)^n, & k < k_I, \\ \hat{v}(k) - C_I, & k \geqslant k_I. \end{cases}$$

Clearly, $\tilde{w}$ is C^1 on $(0, \infty)$, C^2 on $(0, k_I) \cup (k_I, \infty)$, and, according to (7.8), is a solution to the Bellman equation

$$\min\{r\tilde{w} - \mathscr{L}\tilde{w}, \tilde{w} + C_I - \hat{v}\} = 0.$$

A standard verification argument will lead to $\tilde{w} = w$, with an optimal stopping time given by $\tau_I^* = \inf\{t \geqslant 0 : \hat{K}_t \geqslant k_I\}$.

Step 4: It remains to check the uniqueness of the solution in $(0, \infty)$ to (7.7). Consider then the function defined on $(0, \infty)$ by $G(k) = \hat{v}(k) - (k/n)\hat{v}'(k)$. We have $G(0^+) = \hat{v}(0^+) < C_I$, and $G(k) \sim kp(1-\lambda)(1-1/n)$ as k goes to infinity, so that G goes to infinity as k goes to infinity. Moreover, since $G'(k) = (1-1/n)\hat{v}'(k) - k/n\hat{v}''(k) > 0$ by the strict increasing monotonicity and concavity of $\hat{v}$, we obtain the required result, i.e. the existence and uniqueness of a solution k_I in $(0, \infty)$ to $G(k_I) = C_I$. $\square$

Next, we characterize the position of k_I with respect to k_b and k_s, depending on the entry cost C_I, and analyze the monotonicity of k_I with respect to C_I.

Proposition 7.1. *k_I is strictly increasing with respect to C_I. Moreover,*

- *$k_I < k_b$ if and only if $C_I < (n-1)/npk_b + \hat{v}(0^+)$. In this case, we have*

$$k_I = \frac{n}{n-1} \frac{C_I - \hat{v}(0^+)}{p}. \tag{7.9}$$

- *$k_I > k_s$ if and only if $C_I > (n-1)/np(1-\lambda)k_s + C_1$. In this case, we have*

$$k_I = \frac{n}{n-1} \frac{C_I - C_1}{p(1-\lambda)}. \tag{7.10}$$

- *$k_I \in [k_b, k_s]$ if and only if $(n-1)/npk_b + \hat{v}(0^+) \leqslant C_I \leqslant (n-1)/np(1-\lambda)k_s + C_1$.*

Proof. (1) We recall from Theorem 7.2 that k_I is the unique solution in $(0, \infty)$ to Eq. (7.7): $G(k_I) = C_I$. We have already seen in the proof of this last theorem that G is strictly increasing. This means that k_I is strictly increasing with C_I.

(2) From expression (6.5) of $\hat{v} \in (0, k_b)$, we deduce that k_I solution to $G(k_I) = C_I$ lies in $(0, k_b)$ if and only if

$$k_I = \frac{n}{n-1} \frac{C_I - \hat{v}(0^+)}{p} < k_b. \tag{7.11}$$

The above bound is satisfied iff $C_I < (n-1)/npk_b + \hat{v}(0^+)$.

The case where $k_I > k_s$ can be studied similarly. And finally, the last case is a consequence of the two previous ones. $\square$

7.2. Optimal policies and entry decision

Let us first recall the following Skorohod Lemma.

Lemma 7.4. *For any finite a.s. stopping time τ_I, any initial state $k \geqslant 0$ and given two boundaries $0 \leqslant k_b < k_s < \infty$, there exist unique cadlag adapted processes K^* and nondecreasing processes $(L^*, M^*) \in \mathscr{P}(\tau_I)$ satisfying the following Skorohod problem*

$\mathscr{S}(\tau_I, k, k_b, k_s)$:

$$\mathrm{d}K_t^* = K_t^*(\delta\,\mathrm{d}t + \gamma\,\mathrm{d}W_t) + \mathrm{d}L_t^* - \mathrm{d}M_t^*, \quad t \geqslant 0, \quad K_{0^-}^* = k, \tag{7.12}$$

$$K_t^* \in [k_b, k_s] \quad a.e., \ t \geqslant \tau_I, \tag{7.13}$$

$$\int_{\tau_I}^{\infty} 1_{K_u^* > k_b}\,\mathrm{d}L_u^* = 0, \tag{7.14}$$

$$\int_{\tau_I}^{\infty} 1_{K_u^* < k_s}\,\mathrm{d}M_u^* = 0. \tag{7.15}$$

Moreover, if $K_{\tau_I^-}^*(\omega) \in [k_b, k_s]$, *then* $L^*(\omega)$ *and* $M^*(\omega)$ *are continuous. When* $K_{\tau_I^-}^*(\omega) < k_b$, $L_{\tau_I}^*(\omega) = k_b - K_{\tau_I^-}^*(\omega)$, $M_{\tau_I}^*(\omega) = 0$ *and* $K_{\tau_I}^*(\omega) = k_b$, *and when* $K_{\tau_I^-}^*(\omega) > k_s$, $L_{\tau_I}^*(\omega) = 0$, $M_{\tau_I}^*(\omega) = K_{\tau_I^-}^*(\omega) - k_s$ *and* $K_{\tau_I}^*(\omega) = k_s$.

Remark 7.2. When $\tau_I = 0$, the previous lemma is well known and proven (cf. [10]). A straightforward adaptation shows the result for an a.s. finite stopping time τ_I. The solution K^* to the above equations is a reflected diffusion process at the boundaries k_b and k_s, and the processes L^* and M^* are the local times of K^* at k_b and k_s. Conditions (7.14) and (7.15) imply that L^*, (resp. M^*), increases only when K^* hits the boundary k_b (resp. k_s). It is also known that the r-potential of L^* and M^* is finite, i.e., $E[\int_0^\infty \mathrm{e}^{-rt}\,\mathrm{d}L_t^*] < \infty$, and $E[\int_0^\infty \mathrm{e}^{-rt}\,\mathrm{d}M_t^*] < \infty$ (cf. [14, Chapter X]). Therefore, the control (L^*, M^*) is admissible, i.e., $(L^*, M^*) \in \mathscr{A}_{\tau_I}(k)$.

Theorem 7.3. *For* $k \geqslant 0$, *let* (K^*, L^*, M^*) *be the solution to the Skorohod problem* $\mathscr{S}(\tau_I^*, k, k_b, k_s)$, *where* τ_I^* *is the optimal stopping time in Theorem* 7.2. *Then, we have*

$$v(k) = w(k) = J(k, L^*, M^*, \tau_I^*), \quad k \geqslant 0.$$

Proof. *Step* 1: Let τ_I be an a.s. finite stopping time, $(L, M) \in \mathscr{A}_{\tau_I}(k)$, K the associated capital solution to (2.2). By applying Itô's formula to $\mathrm{e}^{-rt}\hat{v}(K_t)$ between τ_I and τ_n, where $\tau_n = \inf\{t \geqslant \tau_I : K_t \geqslant n\} \wedge n$, we obtain by the same arguments as in the proof of Proposition 4.1

$$\mathrm{e}^{-r\tau_I}\hat{v}(K_{\tau_I^-}) \geqslant E\left[\int_{\tau_I}^{\tau_n} \mathrm{e}^{-rt}(\Pi(K_t)\,\mathrm{d}t - p\,\mathrm{d}L_t + (1-\lambda)p\,\mathrm{d}M_t)\middle|\mathscr{F}_{\tau_I}\right].$$

Since $\hat{K}_{\tau_I} = K_{\tau_I^-}$, we get

$$\begin{aligned} w(k) &\geqslant E[\mathrm{e}^{-r\tau_I}(\hat{v}(\hat{K}_{\tau_I}) - C_I)] \\ &\geqslant E\left[\int_{\tau_I}^{\tau_n} \mathrm{e}^{-rt}(\Pi(K_t)\,\mathrm{d}t - p\,\mathrm{d}L_t + (1-\lambda)p\,\mathrm{d}M_t) - \mathrm{e}^{-r\tau_I}C_I\right]. \end{aligned}$$

Therefore, $w(k) \geqslant J(k, L, M, \tau_I)$ by letting $n \to \infty$. We obtain the result for any τ_I in $\mathscr{T}$ by truncating with $\tau_I \wedge T$, and sending T to infinity. From the arbitrariness of (L, M, τ_I), it is clear that $w(k) \geqslant v(k)$.

Step 2: From (7.13), the reflected process K^* lies a.s. in $[k_b, k_s]$ for $t \geqslant \tau_I^*$ so that

$$r\hat{v}(K_t^*) - \mathscr{L}\hat{v}(K_t^*) - \Pi(K_t^*) = 0, \quad t \geqslant \tau_I^* \quad \text{a.s.}$$

By applying Itô's formula to $e^{-rt}\hat{v}(K_t^*)$ as in Step 1, we get

$$E[e^{-r\tau_n}\hat{v}(K_{\tau_n}^*)] = E[e^{-r\tau_I^*}\hat{v}(\hat{K}_{\tau_I^*})] - E\left[\int_{\tau_I^*}^{\tau_n} e^{-rt}\Pi(K_t^*)\,dt\right] + E\left[\int_{\tau_I^*}^{\tau_n} e^{-rt}\hat{v}'(K_t^*)(dL_t^* - dM_t^*)\right], \tag{7.16}$$

where $\tau_n = \inf\{t \geqslant \tau_I^* : K_t^* \geqslant n\} \wedge n$. Now, in view of (7.14) and (7.15), we have

$$E\left[\int_{\tau_I^*}^{\tau_n} e^{-rt}\hat{v}'(K_t^*)(dL_t^* - dM_t^*)\right]$$
$$= E\left[\int_{\tau_I^*}^{\tau_n} e^{-rt}\hat{v}'(K_t^*)(1_{K_t^*=k_b}\,dL_t^* - 1_{K_t^*=k_s}\,dM_t^*)\right]$$
$$= E\left[\int_{\tau_I^*}^{\tau_n} e^{-rt}(p\,dL_t^* - p(1-\lambda)\,dM_t^*)\right],$$

since $\hat{v}'(k_b) = p$ and $\hat{v}'(k_s) = p(1-\lambda)$. Plugging into (7.16) yields

$$w(k) = E[e^{-r\tau_I^*}(\hat{v}(\hat{K}_{\tau_I^*}) - C_I)]$$
$$= E[e^{-r\tau_n}\hat{v}(K_{\tau_n}^*)] + E\left[\int_{\tau_I^*}^{\tau_n} e^{-rt}\Pi(K_t^*)\,dt\right] + E\left[\int_{\tau_I^*}^{\tau_n} e^{-rt}(-p\,dL_t^* + p(1-\lambda)\,dM_t^*) - e^{-r\tau_I^*}C_I\right]. \tag{7.17}$$

Note that the first two terms in the RHS of (7.17) are bounded since K^* lies a.e. in the compact set $[k_b, k_s]$. Since $E[\int_0^\infty e^{-rt}\,dL_t^*] < \infty$, and $E[\int_0^\infty e^{-rt}\,dM_t^*] < \infty$, we can send n to infinity, and obtain by the dominated convergence theorem $w(k) = J(k, L^*, M^*, \tau_I^*)$, which combined with the inequality of Step 1, completes the proof. □

8. Economic interpretation and conclusions

The main results in this paper (Theorems 6.1, 7.2, and 7.3) provide a complete and explicit solution to an optimal partially reversible investment with entry decisions. It mathematically validates several economic intuitions.

- A company will decide to activate its production once a level of capital k_I is reached. From that time, the company will increase or reduce investment in order to maintain its production capacity in a closed bounded interval $[k_b, k_s]$. The value function and the boundaries k_I, k_b, and k_s can be computed quite explicitly.

- The entry-threshold k_I is increasing with respect to the entry cost.
- The expansion threshold k_b decreases with respect to the conversion factor p. This is in accordance with the intuition that the more expensive the expansion, the smaller window of chances which means the smaller expansion region $[0, k_b]$.
- For a fixed p, when λ is decreasing, then k_s is decreasing. This confirms that the intuition the contraction region $[k_s, \infty)$ should be increasing with a higher contraction ratio. Moreover, when λ goes to zero, i.e. expansion and contraction ratios are close, k_s converges to k_b, while in the opposite extreme case when λ goes to 1, i.e. contraction factor goes to zero, k_s goes to infinity.

Our analysis is carried out in a one-dimensional context, and relies crucially on the concavity of the auxiliary value function that in turn implies its smoothness. This is a result of a compromise between complicated economic issues and mathematical tractability. We admit that an even more realistic and general model should be a multidimensional one that incorporates, for instance, stochastic variable inputs in the production function other than the production capacity, and a nondeterministic cost function. This would be an interesting direction for future research.

Appendix A. Proof of dynamic programming principle (5.1)

We recall the notation: for any $k>0$ and $(L, M) \in \mathscr{A}_0(k)$,

$$\hat{J}(k, L, M) = E\left[\int_0^\infty e^{-rt}(\Pi(K_t)\,dt - p\,dL_t + (1-\lambda)p\,dM_t)\right],$$

so that

$$\hat{v}(k) = \sup_{(L,M)\in\mathscr{A}_0(k)} J(k, L, M).$$

(1) For any $k>0$, $(L, M) \in \mathscr{A}_0(k)$ and $\theta \in \mathscr{T}$, we have

$$\begin{aligned}\hat{J}(k, L, M) = {} & E\left[\int_0^{\theta^-} e^{-rt}(\Pi(K_t)\,dt - p\,dL_t + (1-\lambda)p\,dM_t)\right] \\ & + E\left[E\left[\int_\theta^\infty e^{-rt}(\Pi(K_t)\,dt - p\,dL_t + (1-\lambda)p\,dM_t)\middle|\mathscr{F}_\tau\right]\right]. \quad (A.1)\end{aligned}$$

Now, by the strong Markov property of K with respect to $\mathbb{F}$, the augmented filtration of the Brownian motion W (for details, see arguments, e.g. in Lemma 3.2 in [17]), we have

$$\begin{aligned} & E\left[\int_\theta^\infty e^{-rt}(\Pi(K_t)\,dt - p\,dL_t + (1-\lambda)p\,dM_t)\middle|\mathscr{F}_\tau\right] \\ & = e^{-r\theta}\hat{J}(K_{\theta^-}, L, M) \leqslant e^{-r\theta}\hat{v}(K_{\theta^-}) \\ & \leqslant e^{-r\theta}(\hat{v}(K_\theta) - p\,\Delta L_\theta + p(1-\lambda)\Delta M_\theta),\end{aligned}$$

where the last inequality follows from Lemma 4.1. Plugging the above inequality into (A.1), we obtain

$$\begin{aligned}
&\hat{J}(k,L,M)\\
&\quad\leqslant E\left[\int_0^{\theta} \mathrm{e}^{-rt}(\Pi(K_t)\,\mathrm{d}t - p\,\mathrm{d}L_t + (1-\lambda)p\,\mathrm{d}M_t) + \mathrm{e}^{-r\theta}\hat{v}(K_\theta)\right]\\
&\quad\leqslant \sup_{(L,M)\in\mathscr{A}_0(k)} E\left[\int_0^{\theta} \mathrm{e}^{-rt}(\Pi(K_t)\,\mathrm{d}t - p\,\mathrm{d}L_t + (1-\lambda)p\,\mathrm{d}M_t) + \mathrm{e}^{-r\theta}\hat{v}(K_\theta)\right].
\end{aligned}$$

Taking the supremum over (L,M) in the LHS of the previous inequality shows that

$$\hat{v}(k)\leqslant \sup_{(L,M)\in\mathscr{A}_0(k)} E\left[\int_0^{\theta} \mathrm{e}^{-rt}(\Pi(K_t)\,\mathrm{d}t - p\,\mathrm{d}L_t + (1-\lambda)p\,\mathrm{d}M_t) + \mathrm{e}^{-r\theta}\hat{v}(K_\theta)\right].$$

(2) Conversely, given an arbitrary $(L,M)\in\mathscr{A}_0(k)$, $k>0$, $\theta\in\mathscr{T}$, for any $\varepsilon>0$, $\omega\in\Omega$, there exists, by the definition of v, a pair $(L^{\varepsilon,\omega},M^{\varepsilon,\omega})\in\mathscr{A}_\theta(K_{\theta^-}(\omega))$ s.t.

$$v(K_{\theta^-}(\omega)) - \varepsilon < J(K_{\theta^-}(\omega), L^{\varepsilon,\omega}, M^{\varepsilon,\omega}). \tag{A.2}$$

Then, consider the processes $(\tilde{L},\tilde{M})$ defined for all $t\geqslant 0$ and $\omega\in\Omega$ by

$$(\tilde{L}_t,\tilde{M}_t)(\omega) = (L_t,M_t)(\omega)1_{t<\theta}(\omega) + (L_t^{\varepsilon,\omega}+L_{\theta^-}, M_t^{\varepsilon,\omega}+M_{\theta^-})(\omega)1_{t\geqslant\theta}(\omega).$$

The technical point here is to prove that $(\tilde{L},\tilde{M})$ is adapted and this can be done by a measurable selection result as, e.g. in [1] or [2]. Then, the process $(\tilde{L},\tilde{M})\in\mathscr{A}_0(k)$ and we denote by $\tilde{K}$ the associated controlled process solution to (2.2). From the pathwise uniqueness for the SDE (2.2), we have a.s. $\tilde{K}_t = K_t^{\varepsilon}$ for $t\geqslant\theta$, where K^{ε} is the solution to

$$\begin{aligned}
&\mathrm{d}K_t^{\varepsilon} = K_t^{\varepsilon}(\delta\,\mathrm{d}t + \gamma\,\mathrm{d}W_t) + \mathrm{d}L_t^{\varepsilon} - \mathrm{d}M_t^{\varepsilon}, \quad t\geqslant\theta,\\
&K_{\theta^-}^{\varepsilon} = K_{\theta^-}.
\end{aligned}$$

We then have

$$\begin{aligned}
\hat{v}(k) &\geqslant \hat{J}(k,\tilde{L},\tilde{M})\\
&= E\left[\int_0^{\theta^-} \mathrm{e}^{-rt}(\Pi(K_t)\,\mathrm{d}t - p\,\mathrm{d}L_t + (1-\lambda)p\,\mathrm{d}M_t)\right]\\
&\quad + E\left[E\left[\int_\theta^{\infty} \mathrm{e}^{-rt}(\Pi(K_t^{\varepsilon})\,\mathrm{d}t - p\,\mathrm{d}L_t^{\varepsilon} + (1-\lambda)p\,\mathrm{d}M_t^{\varepsilon})\middle|\mathscr{F}_\tau\right]\right]\\
&= E\left[\int_0^{\theta^-} \mathrm{e}^{-rt}(\Pi(K_t)\,\mathrm{d}t - p\,\mathrm{d}L_t + (1-\lambda)p\,\mathrm{d}M_t) + \mathrm{e}^{-r\theta}J(K_{\theta^-},L^{\varepsilon},M^{\varepsilon})\right]\\
&\geqslant E\left[\int_0^{\theta^-} \mathrm{e}^{-rt}(\Pi(K_t)\,\mathrm{d}t - p\,\mathrm{d}L_t + (1-\lambda)p\,\mathrm{d}M_t) + \mathrm{e}^{-r\theta}v(K_{\theta^-})\right] - \varepsilon\\
&\geqslant E\left[\int_0^{\theta} \mathrm{e}^{-rt}(\Pi(K_t)\,\mathrm{d}t - p\,\mathrm{d}L_t + (1-\lambda)p\,\mathrm{d}M_t) + \mathrm{e}^{-r\theta}v(K_\theta)\right] - \varepsilon,
\end{aligned}$$

where the first inequality follows from (A.2) and the second one from Lemma 4.1. Since (L, M) is arbitrary in $\mathscr{A}_0(k)$, by sending ε to zero, we obtain

$$\hat{v}(k) \geqslant \sup_{(L,M)\in\mathscr{A}_0(k)} E\left[\int_0^\theta \mathrm{e}^{-rt}(\Pi(K_t)\,\mathrm{d}t - p\,\mathrm{d}L_t + (1-\lambda)p\,\mathrm{d}M_t) + \mathrm{e}^{-r\theta}\hat{v}(K_\theta)\right]$$

and this ends the proof. □

Appendix B. Proof of Theorem 5.1

B.1. Supersolution Viscosity Property

Fix some $k_0 > 0$ and C^2 function φ such that $\hat{v}(k_0) = \varphi(k_0)$ and $\varphi(k) \leqslant \hat{v}(k)$ for all k in a neighborhood $\bar{B}_\varepsilon(k_0) = [k_0 - \varepsilon, k_0 + \varepsilon]$ of k_0 $(0 < \varepsilon < k_0)$. Consider the admissible control $(L, M) \in \mathscr{A}_0(k_0)$ defined by $M = 0$ and

$$L_t = \begin{cases} 0, & t = 0, \\ \eta, & t \geqslant 0, \end{cases}$$

where $0 \leqslant \eta < \varepsilon$. Define the exit time $\tau_\varepsilon = \inf\{t \geqslant 0, K_t \notin \bar{B}_\varepsilon(x_0)\}$. Here K is the production capacity starting from k_0 and controlled by (L, M) above. Note that K has at most one jump at $t = 0$ and is continuous on $(0, \tau_\varepsilon]$. By the dynamic programming principle (5.1) with $\theta = \tau_\varepsilon \wedge h$, $h > 0$, we have

$$\begin{aligned}\varphi(k_0) = \hat{v}(k_0) &\geqslant E\left[\int_0^{\tau_\varepsilon\wedge h} \mathrm{e}^{-rt}(\Pi(K_t)\,\mathrm{d}t - p\,\mathrm{d}L_t) + \mathrm{e}^{-r(\tau_\varepsilon\wedge h)}\hat{v}(K_{\tau_\varepsilon\wedge h})\right] \\ &\geqslant E\left[\int_0^{\tau_\varepsilon\wedge h} \mathrm{e}^{-rt}(\Pi(K_t)\,\mathrm{d}t - p\,\mathrm{d}L_t) + \mathrm{e}^{-r(\tau_\varepsilon\wedge h)}\varphi(K_{\tau_\varepsilon\wedge h})\right]. \qquad \text{(B.1)}\end{aligned}$$

Applying Itô's formula to the process $\mathrm{e}^{-rt}\varphi(K_t)$ between 0 and $\tau_\varepsilon \wedge h$, and taking expectation, we obtain similarly as in the proof of Proposition 4.1 by noting also that $\mathrm{d}L_t^{\mathrm{c}} = 0$

$$\begin{aligned}E[\mathrm{e}^{-r(\tau_\varepsilon\wedge h)}\varphi(K_{\tau_\varepsilon\wedge h})] = \varphi(k_0) &+ E\left[\int_0^{\tau_\varepsilon\wedge h} \mathrm{e}^{-rt}(-r\varphi + \mathscr{L}\varphi)(K_t)\,\mathrm{d}t\right] \\ &+ E\left[\sum_{0\leqslant t\leqslant\tau_\varepsilon\wedge h} \mathrm{e}^{-rt}[\varphi(K_t) - \varphi(K_{t^-})]\right]. \qquad \text{(B.2)}\end{aligned}$$

Combining relations (B.1) and (B.2), we see

$$\begin{aligned}&E\left[\int_0^{\tau_\varepsilon\wedge h} \mathrm{e}^{-rt}(r\varphi - \mathscr{L}\varphi - \Pi)(K_t)\,\mathrm{d}t\right] + E\left[\int_0^{\tau_\varepsilon\wedge h} \mathrm{e}^{-rt}p\,\mathrm{d}L_t\right] \\ &\quad - E\left[\sum_{0\leqslant t\leqslant\tau_\varepsilon\wedge h} \mathrm{e}^{-rt}[\varphi(K_t) - \varphi(K_{t^-})]\right] \geqslant 0. \qquad \text{(B.3)}\end{aligned}$$

- Taking first $\eta = 0$, i.e., $L = 0$, we see that K is continuous, and only the first term in the LHS of (B.3) is nonzero. By dividing the above inequality by h with $h \to 0$, we conclude by the dominated convergence theorem

$$r\varphi(k_0) - \mathscr{L}\varphi(k_0) - \Pi(k_0) \geqslant 0. \tag{B.4}$$

- Now, by taking $\eta > 0$ in (B.3), and noting that L and K jump only at $t = 0$ with size η, we get

$$E\left[\int_0^{\tau_\varepsilon \wedge h} \mathrm{e}^{-rt}(r\varphi - \mathscr{L}\phi - \Pi)(K_t)\,\mathrm{d}t\right] + p\eta - \varphi(k_0 + \eta) + \varphi(k_0) \geqslant 0. \tag{B.5}$$

  Taking $h \to 0$, then dividing by η and letting $\eta \to 0$, we see

$$p - \varphi'(k_0) \geqslant 0. \tag{B.6}$$

- On the other hand, taking an admissible control $(L, M) \in \mathscr{A}_0(k_0)$ such that $L = 0$ and

$$M_t = \begin{cases} 0, & t = 0, \\ \eta > 0, & t \geqslant 0 \end{cases}$$

  and going over a similar argument, inequality (B.5) is replaced by

$$E\left[\int_0^{\tau_\varepsilon \wedge h} \mathrm{e}^{-rt}(r\varphi - \mathscr{L}\phi - \Pi)(K_t)\,\mathrm{d}t\right] - p(1-\lambda)\eta - \varphi(k_0 - \eta) + \varphi(k_0) \geqslant 0.$$

  Taking $h \to 0$, then dividing by η and letting $\eta \to 0$, we obtain

$$-p(1-\lambda) + \varphi'(k_0) \geqslant 0. \tag{B.7}$$

  This proves the required supersolution viscosity property

$$\min\{r\varphi(k_0) - \mathscr{L}\varphi(k_0) - \Pi(k_0), -\varphi'(k_0) + p, \varphi'(k_0) - p(1-\lambda)\} \geqslant 0. \tag{B.8}$$

B.2. Subsolution Viscosity Property

We prove this part by contradiction. Suppose the claim is not true. Then, there exist $k_0 > 0$, $0 < \varepsilon < k_0$, a C^2 function φ with $\varphi(k_0) = \hat{v}(k_0)$ and $\varphi \geqslant \hat{v}$ in $\bar{B}_\varepsilon(k_0) = [k_0 - \varepsilon, k_0 + \varepsilon]$, and $v > 0$ such that for all $k \in \bar{B}_\varepsilon(k_0)$

$$r\varphi(k) - \mathscr{L}\varphi(k) - \Pi(k) \geqslant v, \tag{B.9}$$

$$p(1-\lambda) + v \leqslant \varphi'(k) \leqslant p - v. \tag{B.10}$$

Given any admissible control $(L, M) \in \mathscr{A}_0(k_0)$, consider the exit time $\tau_\varepsilon = \inf\{t \geqslant 0, K_t \notin B_\varepsilon(x_0)\}$. (Here K is the production capacity starting from k_0 and controlled by (L, M).) By applying Itô's formula to $\mathrm{e}^{-rt}\varphi(K_t)$, we get

$$\begin{aligned} E[\mathrm{e}^{-r\tau_\varepsilon}\varphi(K_{\tau_\varepsilon^-})] &= \varphi(k_0) + E\left[\int_0^{\tau_\varepsilon} \mathrm{e}^{-rt}(-r\varphi + \mathscr{L}\varphi)(K_t)\,\mathrm{d}t\right] \\ &\quad + E\left[\int_0^{\tau_\varepsilon} \mathrm{e}^{-rt}\varphi'(K_t)(\,\mathrm{d}L_t^{\mathrm{c}} - \,\mathrm{d}M_t^{\mathrm{c}})\right] \\ &\quad + E\left[\sum_{0\leqslant t<\tau_\varepsilon} \mathrm{e}^{-rt}[\phi(K_t) - \phi(K_{t^-})]\right]. \end{aligned} \tag{B.11}$$

Note that for all $0\leqslant t<\tau_\varepsilon$, $K_t \in \bar{B}_\varepsilon(k_0)$. Then, from Taylor's formula and (B.10), and noting that $\Delta K_t = \Delta L_t - \Delta M_t$, we have for all $0\leqslant t<\tau_\varepsilon$

$$\begin{aligned} \varphi(K_t) - \varphi(K_{t^-}) &= \Delta K_t \int_0^1 \varphi'(K_t + z\,\Delta K_t)\,\mathrm{d}z \\ &\leqslant (p - v)\Delta L_t - (p(1-\lambda) + v)\Delta M_t. \end{aligned} \tag{B.12}$$

In light of relations (B.9)–(B.12), we thus obtain

$$\begin{aligned} E[\mathrm{e}^{-r\tau_\varepsilon}\varphi(K_{\tau_\varepsilon^-})] &\leqslant \varphi(k_0) + E\left[\int_0^{\tau_\varepsilon} \mathrm{e}^{-rt}(-\Pi - v)(K_t)\,\mathrm{d}t\right] + E\left[\int_0^{\tau_\varepsilon^-} \mathrm{e}^{-rt}(p - v)\,\mathrm{d}L_t\right] \\ &\quad - E\left[\int_0^{\tau_\varepsilon^-} \mathrm{e}^{-rt}(p(1-\lambda) + v)\,\mathrm{d}M_t\right] \\ &= \varphi(k_0) + E\left[\int_0^{\tau_\varepsilon} \mathrm{e}^{-rt}(-\Pi(K_t)\,\mathrm{d}t + p\,\mathrm{d}L_t - p(1-\lambda)\,\mathrm{d}M_t)\right] \\ &\quad - E[\mathrm{e}^{-r\tau_\varepsilon}p\,\Delta L_{\tau_\varepsilon}] + E[\mathrm{e}^{-r\tau_\varepsilon}p(1-\lambda)\Delta M_{\tau_\varepsilon}] \\ &\quad - v\left\{E\left[\int_0^{\tau_\varepsilon} \mathrm{e}^{-rt}\,\mathrm{d}t\right] + E\left[\int_0^{\tau_\varepsilon^-} \mathrm{e}^{-rt}(\mathrm{d}L_t + \mathrm{d}M_t)\right]\right\}. \end{aligned} \tag{B.13}$$

Note that while $K_{\tau_\varepsilon^-} \in \bar{B}_\varepsilon(k_0)$, K_{τ_ε} is either on the boundary $\partial B_\varepsilon(k_0)$ or out of $\bar{B}_\varepsilon(k_0)$. However, there is some random variable α valued in $[0, 1]$ such that

$$\begin{aligned} k_\alpha &= K_{\tau_\varepsilon^-} + \alpha\,\Delta K_{\tau_\varepsilon} \\ &= K_{\tau_\varepsilon^-} + \alpha(\Delta L_{\tau_\varepsilon} - \Delta M_{\tau_\varepsilon}) \in \partial\bar{B}_\varepsilon(k_0) = \{k_0 - \varepsilon, k_0 + \varepsilon\}. \end{aligned}$$

Then, similarly as in (B.12), we have

$$\varphi(k_\alpha) - \varphi(K_{\tau_\varepsilon^-}) \leqslant \alpha(p - v)\Delta L_{\tau_\varepsilon} - \alpha(p(1-\lambda) + v)\Delta M_{\tau_\varepsilon}. \tag{B.14}$$

Note that $K_{\tau_\varepsilon} = k_\alpha + (1-\alpha)(\Delta L_{\tau_\varepsilon} - \Delta M_{\tau_\varepsilon})$, and so from Lemma 4.1

$$\hat{v}(k_\alpha) \geqslant -p(1-\alpha)\Delta L_{\tau_\varepsilon} + p(1-\lambda)(1-\alpha)\Delta M_{\tau_\varepsilon} + \hat{v}(K_{\tau_\varepsilon}). \tag{B.15}$$

Recalling that $\varphi(k_\alpha) \geqslant \hat{v}(k_\alpha)$, inequalities (B.14) and (B.15) imply

$$\varphi(K_{\tau_\varepsilon^-}) \geqslant \hat{v}(K_{\tau_\varepsilon}) - (p - \alpha v)\Delta L_{\tau_\varepsilon} + (p(1-\lambda) + \alpha v)\Delta M_{\tau_\varepsilon}.$$

Plugging this last inequality into (B.13) and recalling that $\varphi(k_0)=\hat{v}(k_0)$ yields

$$\begin{aligned}\hat{v}(k_0) \geqslant & E\left[\int_0^{\tau_\varepsilon} \mathrm{e}^{-rt}(\Pi(K_t)\,\mathrm{d}t - p\,\mathrm{d}L_t + p(1-\lambda)\,\mathrm{d}M_t) + \hat{v}(K_{\tau_\varepsilon})\right] \\ & + v\left\{E\left[\int_0^{\tau_\varepsilon} \mathrm{e}^{-rt}\,\mathrm{d}t\right] + E\left[\int_0^{\tau_\varepsilon^-} \mathrm{e}^{-rt}(\mathrm{d}L_t + \mathrm{d}M_t)\right]\right. \\ & \left. + E[\mathrm{e}^{-r\tau_\varepsilon}\alpha(\Delta L_{\tau_\varepsilon} + \Delta M_{\tau_\varepsilon})]\right\}. \end{aligned} \tag{B.16}$$

- We now claim that there exists a constant $g_0 > 0$ such that for all $(L,M)\in\mathscr{A}_0(k_0)$

$$E\left[\int_0^{\tau_\varepsilon} \mathrm{e}^{-rt}\,\mathrm{d}t\right] + E\left[\int_0^{\tau_\varepsilon^-} \mathrm{e}^{-rt}(\mathrm{d}L_t + \mathrm{d}M_t)\right] + E[\mathrm{e}^{-r\tau_\varepsilon}\alpha(\Delta L_{\tau_\varepsilon} + \Delta M_{\tau_\varepsilon})] \geqslant g_0. \tag{B.17}$$

Indeed, one can always find some constant $G_0 > 0$ such that the C^2 function

$$\psi(k) = G_0((k-k_0)^2 - \varepsilon^2),$$

satisfies

$$\min\{r\psi - \mathscr{L}\psi + 1, 1 - |\psi'|\} \geqslant 0, \quad \text{on } \bar{B}_\varepsilon(k_0),$$

$$\psi = 0, \quad \text{on } \partial\bar{B}_\varepsilon(k_0).$$

For instance, we can choose

$$G_0 = \min\left\{\frac{1}{r\varepsilon^2 + 2\varepsilon|\delta|(k_0+\varepsilon) + \gamma^2(k_0+\varepsilon)^2}, \frac{1}{2\varepsilon}\right\} > 0.$$

By applying again Itô's lemma, we then get

$$E[\mathrm{e}^{-r\tau_\varepsilon}\psi(K_{\tau_\varepsilon^-})] \leqslant \psi(k_0) + E\left[\int_0^{\tau_\varepsilon} \mathrm{e}^{-rt}\,\mathrm{d}t\right] + E\left[\int_0^{\tau_\varepsilon^-} \mathrm{e}^{-rt}(\,\mathrm{d}L_t + \,\mathrm{d}M_t)\right]. \tag{B.18}$$

Since $\psi'(k) \geqslant -1$, we have

$$\begin{aligned}\psi(K_{\tau_\varepsilon^-}) - \psi(k_\alpha) &\geqslant -(K_{\tau_\varepsilon^-} - k_\alpha) = \alpha(\Delta L_{\tau_\varepsilon} - \Delta M_{\tau_\varepsilon}) \\ &\geqslant -\alpha\,\Delta M_{\tau_\varepsilon}.\end{aligned}$$

Plugging into (B.18) yields

$$\begin{aligned} & E\left[\int_0^{\tau_\varepsilon} \mathrm{e}^{-rt}\,\mathrm{d}t\right] + E\left[\int_0^{\tau_\varepsilon^-} \mathrm{e}^{-rt}(\mathrm{d}L_t + \mathrm{d}M_t)\right] + E[\mathrm{e}^{-r\tau_\varepsilon}\alpha\,\Delta M_{\tau_\varepsilon}] \\ & \quad \geqslant E[\mathrm{e}^{-r\tau_\varepsilon}\psi(k_\alpha)] - \psi(k_0) = -\psi(k_0) = G_0\varepsilon^2. \end{aligned} \tag{B.19}$$

Hence, claim (B.17) holds with $g_0 = G_0\varepsilon^2$.

- Finally, by taking the supremum over all $(L, M) \in \mathscr{A}_0(k_0)$ in (B.16), and invoking the dynamic programming principle (5.1), we have $\hat{v}(k_0) \geqslant \hat{v}(k_0) + vg_0$, which is the required contradiction.

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