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## Irreversible investment with regime shifts

Xin Guo,<sup>a</sup> Jianjun Miao,<sup>b</sup> and Erwan Morellec<sup>c,d,e,\*</sup>

<sup>a</sup> *Department of Operations Research and Industrial Engineering, Cornell University, 214 Rhodes Hall, Ithaca, NY 14853, USA*

<sup>b</sup> *Department of Economics, Boston University, 270 Bay State Road, Boston, MA 02215, USA*

<sup>c</sup> *Ecole des HEC, University of Lausanne, Lausanne CH 1007, Switzerland*

<sup>d</sup> *International Center FAME, 40bd. du Pont d'Arve, Geneva CH 1211, Switzerland*

<sup>e</sup> *William E. Simon Graduate School of Business Administration, University of Rochester, Rochester NY 14627, USA*

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### Abstract

Under the real options approach to investment under uncertainty, agents formulate optimal policies under the assumption that firms' growth prospects do not vary over time. This paper proposes and solves a model of investment decisions in which the growth rate and volatility of the decision variable shift between different states at random times. A value-maximizing investment policy is derived such that in each regime the firm's investment policy is optimal and recognizes the possibility of a regime shift. Under this policy, investment is intermittent and increases with marginal  $q$ . Moreover, investment typically is very small but, in some states, the capital stock jumps. Implications for marginal  $q$  and the user cost of capital are also examined.

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\*Corresponding author. William E. Simon Graduate School of Business Administration, University of Rochester, Rochester NY 14627, USA.

*E-mail addresses:* [xinguo@orie.cornell.edu](mailto:xinguo@orie.cornell.edu) (X. Guo), [miaoj@bu.edu](mailto:miaoj@bu.edu) (J. Miao), [morellec@simon.rochester.edu](mailto:morellec@simon.rochester.edu) (E. Morellec).

## 1. Introduction

The notion that regime shifts are important in explaining the cyclical features of real macroeconomic variables as proposed by Hamilton [15] is now widely accepted. Motivated by anecdotal evidence, a pervasive manifestation of this view is that regime shifts, by changing firms growth prospects, affect capital accumulation and investment decisions. On economic grounds, there are indeed reasons to believe that regime shifts contain the possibility of significant impact on firms policy choices. For example, business cycle expansion and contraction “regimes” potentially have sizable effects on the profitability or riskiness of investment and, hence, on firms’ willingness to invest in physical or human capital. Yet, despite these potential effects, we still know very little about the relation between regime shifts and investment decisions.

The idea that shifts in a firm’s environment can have first-order effects on its investment policy can be related to the burgeoning literature on investment decisions under uncertainty (see the survey by Dixit and Pindyck [9]). In this literature, investment opportunities are analyzed as options written on real assets and the optimal investment policy is derived by maximizing the value of the option to invest. Because option values depend on the riskiness of the underlying asset, volatility is an important determinant of the optimal investment policy. Despite this observation, models of investment decisions typically presume that this very parameter is fixed. It is not difficult to imagine however that as volatility changes over the business cycle, so does the value-maximizing investment policy.

This paper develops a framework to study the behavior of investment when the dynamics of the decision variable are subject to discrete regime shifts at random times. Following Hamilton, we define shifts in regime for a process as “episodes across which the behavior of the series is markedly different”. To emphasize the impact of regime shifts on investment decisions and capital accumulation, we construct a simple model of capacity choice that builds on earlier work by Pindyck [26] and Abel and Eberly [3]. Specifically, we consider an infinitely lived firm that produces output with its capital stock and variable factors of production. The price of the firm’s output fluctuates randomly, yielding a stochastic continuous stream of cash flows. At any time  $t$ , the firm can (irreversibly) increase capacity by purchasing capital. Investment arises when the marginal valuation of capital equals the purchase price of capital.

Models of investment decisions under uncertainty generally presume that the firm’s operating profits are subject to a multiplicative shock that evolves according to a geometric Brownian motion.<sup>1</sup> Implicit in this modeling is the assumption that the

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<sup>1</sup>Statistical tests of the option theory of irreversible investment typically are specified under this assumption (see for example [18]). In fact, Harchaoui and Lasserre note that “the empirical experiment in which agents respond to changes in  $\alpha$  [the drift rate] or  $\sigma$  [volatility]” cannot be experimented within their econometric model because the theoretical model does not yield any analytical solution for this underlying process. In this paper, we provide such a solution. In his survey paper, Chirinko [8, pp. 1905–1906] also points out the importance of the time-varying volatility for the econometric specification of investment equations.

firm's growth prospects do not vary over time. This paper solves for the value-maximizing investment policy when the growth rate and volatility of the marginal revenue product of capital are subject to discrete regime shifts. The analysis demonstrates that, in contrast with standard models of investment, the optimal decision rule is not described by a simple threshold for the marginal revenue product of capital. Instead, the optimal investment policy is characterized by a different threshold for each regime. Moreover, because of the possibility of a regime shift, the value-maximizing threshold in each regime reflects the possibility for the firm to invest in the other regimes. That is, a value-maximizing policy is derived such that in each regime the firm's investment policy is optimal, conditional on the optimal investment policy in the other regimes.

An important question is whether regime shifts actually affect growth and capital accumulation. To answer this question, we examine the implications of the model for the optimal rate of investment. These implications are generally consistent with recent evidence on firms' investment behavior (see [1] or [7]). In particular, the model predicts that investment is intermittent and increasing with marginal  $q$ . Moreover, the state space of the dynamic investment problem can be partitioned into various domains including an *inaction region* where no investment occurs. Outside of this region, the optimal rate of investment can be in one of two regimes: infinitesimal or lumpy. Investment is infinitesimal at the investment threshold. Investment is lumpy in the *transient region* and at the initial date if the state of the system is *in the action region*. Also, while it is always optimal to invest in the action region, the optimality of investment is regime dependent in the transient region. That is, regime shifts generate some time-series variation in the present value of future cash flows to current cash flows that may induce the firm to invest following a regime shift.

The analysis in the present paper relates to two different strands of literature. First, from an economic point of view, it relates to the investment literature that combines real options features—irreversibility and a continuous stochastic process—with neoclassical features—no indivisibilities. In these models, investment is intermittent and, in the absence of fixed adjustment costs, involves marginal adjustments in the stock of capital (see [26,2,6,13]). When fixed adjustment costs are introduced, investment is intermittent and lumpy, and the optimal policy involves impulse control techniques (see [4] or [7]). In the present paper, there are no fixed adjustment costs. Yet, the optimal investment policy involves both marginal adjustments and jumps in the stock of capital.

From a technical view point, the present paper relates to a series of recent papers on option pricing with regime shifts (see [15,16,10]). One of our main contributions is the extension of techniques in these papers to the case of stochastic control problems where control policies change the underlying diffusion process. In particular, we use the solution to the optimal stopping problem derived by Guo [15] to analyze the recurrent investment decision of a firm with divisible capital. Because the firm's problem is homogeneous, the recurrent model displays a structure that is similar to the stopping problem except that the firm obtains a new investment option whenever it stops.

The paper that is most closely related to the present analysis is Driffill and Sola [11]. These authors also analyze investment decisions when the dynamics of the state variable can shift between several regimes. One essential difference between the two papers is that we examine capacity choice and the valuation of interrelated options whereas they focus on the valuation of a single investment opportunity (in the spirit of McDonald and Siegel [24]). Another important point of departure is that we solve our model analytically whereas they solve their model numerically. Finally, we derive implications for capital accumulation, marginal  $q$ , and the user cost of capital, which are not examined in their paper.

The remainder of the paper is organized as follows. Section 2 presents the basic model of investment decisions with regime shifts. Section 3 derives the firm's objective function and optimality conditions. Section 4 determines the value-maximizing investment policy. Section 5 presents simulation results. Section 6 investigates the implications of the optimal investment policy for capital accumulation and growth. Section 7 analyses marginal  $q$  and the user cost of capital. Section 8 concludes.

## 2. The model

This paper provides an analysis of investment decisions under uncertainty when the dynamics of the state variable shift between different states at random times. Throughout the paper, agents are risk neutral and discount cash flows at a constant rate  $\rho$ . Time is continuous and uncertainty is modeled by a complete probability space  $(\Omega, \mathcal{F}, P)$ . For any process  $(y_t)_{t \geq 0}$  defined on  $(\Omega, \mathcal{F}, P)$ ,  $\mathcal{F}^y = (\mathcal{F}_t^y)_{t \geq 0}$  denotes the  $P$ -augmentation of the filtration  $(\sigma(y_s; s \leq t))_{t \geq 0}$  generated by  $y$ .

*Technology:* Consider an infinitely-lived firm that produces output with its capital stock and variable factors of production. Assume for simplicity that the good produced by the firm is not storable so that output equals demand. In addition, suppose that the firm's capital stock depreciates at a constant exponential rate  $\delta \geq 0$  and that its operating profit is given by a linearly homogenous function  $\pi: \mathbb{R}_{++} \times \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$  satisfying:

$$\pi(x_t, k_t) = \frac{1}{1-\alpha} x_t^\alpha k_t^{1-\alpha}, \quad \alpha \in (0, 1), \quad (1)$$

where  $(k_t)_{t \geq 0}$  is a nonnegative process representing the firm's capital stock, and  $(x_t)_{t \geq 0}$  is a demand shock with law of motion specified below. As shown by Abel and Eberly [3] and Morellec [25], Eq. (1) is consistent with a price taking firm whose technology exhibits decreasing returns to scale or with a monopolist facing a constant returns to scale technology and a constant elasticity demand curve.

At any time  $t$ , the firm can increase capacity by purchasing capital at the price  $p$ . The capital input is homogenous and perfectly divisible and the firm is a price-taker in the market for capital goods. The optimality of the decision to invest depends on the incremental profits associated with an increase in the capital stock and the price of capital. It also depends on other dimensions of the firm's environment such as

ongoing uncertainty in profits or the firm’s ability to reverse its decisions. Following Pindyck [26], Abel and Eberly [3], and Grenadier [14], we consider that investment is irreversible.<sup>2</sup> In contrast to these studies, we do not assume that  $(x_t)_{t \geq 0}$  is governed by a Markov process with constant drift and volatility but instead characterize capital accumulation and investment decisions when the dynamics of the demand shock shift between different states at random times. As shown below, this specification introduces some interesting, yet tractable, variations in the firm’s growth prospects.

*Dynamics of the demand shock:* Throughout the paper, the dynamics of the demand shock  $(x_t)_{t \geq 0}$  are governed by a Markov regime switching model. Within the current setting, such a model may reflect the impact of the business cycle on the cash flows generated by the firm’s assets. Depending on the state of the economy, the dynamics of the demand shift parameter for the good produced by the firm shift from one state to another, in turn changing the dynamics—growth rate and volatility—of the firm cash flows.

Specifically, we presume that the dynamics of  $(x_t)_{t \geq 0}$  can shift between two states and are governed by the process:<sup>3</sup>

$$dx_t = \mu_{\varepsilon(t)}x_t dt + \sigma_{\varepsilon(t)}x_t dW_t, \quad x_t > 0, \tag{2}$$

where  $(W_t)_{t \geq 0}$  is standard Brownian motion defined on  $(\Omega, \mathcal{F}, P)$  and  $(\varepsilon_t)_{t \geq 0}$  is a Markov process independent of  $(W_t)_{t \geq 0}$ . The pair  $(\mu_{\varepsilon(t)}, \sigma_{\varepsilon(t)})$  takes different values when the process  $(\varepsilon_t)$  is in different states. For each state  $i$ , there is a known drift parameter  $\mu_i$  and a known volatility parameter  $\sigma_i > 0$ . Moreover, while  $(x_t)_{t \geq 0}$  is not a Markov process,  $(z_t)_{t \geq 0} \equiv (x_t, \varepsilon_t)_{t \geq 0}$  is jointly Markovian if at any time  $t$  the state of  $\varepsilon_t$  is known.

*Regime shifts:* Assume that  $(\varepsilon_t)_{t \geq 0}$  is observable and that the transition probability of  $(\varepsilon_t)_{t \geq 0}$  follows a Poisson law, such that  $(\varepsilon_t)_{t \geq 0}$  is a two-state Markov chain alternating between states 1 and 2. Let  $\lambda_i > 0$  denote the rate of leaving state  $i$  and  $\tau_i$  the time to leave state  $i$ . Within our model, the exponential law holds

$$P(\tau_i > t) = e^{-\lambda_i t}, \quad i = 1, 2 \tag{3}$$

and the process  $\varepsilon(t)$  has the transition matrix between time  $t$  and  $t + \Delta t$ :

$$\begin{bmatrix} 1 - \lambda_1 \Delta t & \lambda_1 \Delta t \\ \lambda_2 \Delta t & 1 - \lambda_2 \Delta t \end{bmatrix}.$$

<sup>2</sup>A natural way to introduce irreversibility within the present model is to consider that capital has no resale value. Because the marginal revenue product of capital is bounded from below by zero, it is never optimal for the firm to sell assets. The model can be extended to consider costly reversibility, abandonment, and the interaction between financing and investment policies. To focus more clearly on the regime shift aspect, we keep these complications out of this paper.

<sup>3</sup>This process has been introduced by Guo [15,16] in a model that addresses the pricing of perpetual lookback options. Our paper extends her analysis to the valuation of multiple interrelated options. Obtaining the exact solution (which is of the viscosity type) to the Hamilton–Jacobi–Bellman equation in our case is analytically more challenging. The nature of the optimal policy is of singular control type, which is similar in spirit to the threshold type stopping rules in [15].

The above set of assumptions captures the idea that both the drift and volatility parameters of the demand shift may change over time at random dates. That is, unlike traditional models of investment, the present model allows for stochastic regime shifts in the parameters of the underlying state variable. In particular, during an infinitesimal time interval  $\Delta t$ , there is a probability  $\lambda_1 \Delta t$  that these parameters shift from  $(\mu_1, \sigma_1)$  to  $(\mu_2, \sigma_2)$  and a probability  $\lambda_2 \Delta t$  that they shift from  $(\mu_2, \sigma_2)$  to  $(\mu_1, \sigma_1)$ .<sup>4</sup>

*Statement of the problem:* The firm's objective is to determine the investment policy that maximizes the present value of future profits net of investment costs. Given the properties of the profit function (1), this investment policy takes the form of a trigger policy that can be described, for every  $\kappa \in [k, +\infty)$  and in each regime  $i$ , by a first passage time of  $(x_t)_{t \geq 0}$  to a constant threshold  $x(\kappa)$ . While the trigger policy is common to previous models of investment decisions under uncertainty, two major differences arise within the present model. First, because the dynamics of the demand shock depend on the current regime, so does the value-maximizing investment policy. In other words, there exists a different trigger threshold  $x_i^*(\kappa)$  for each regime  $i$ . Second, because of the possibility of a regime shift, the optimal trigger threshold in each regime reflects the possibility for the firm to invest in the other regime. That is, the firm has to determine an investment policy in each regime, while taking into the optimal investment policy in the other regime.

### 3. Capacity choice with regime shifts

This section derives optimality conditions for investment when the firm's profit function satisfies Eq. (1) and changes in the demand shock are governed by (2). We start by specifying the firm's objective function.

*Firm's objective function:* The firm's objective is to determine the investment policy that maximizes the expected present value of profits net of investment costs. Following Bertola and Caballero [6], we denote by  $(G_t)_{t \geq 0}$  the right continuous, nonnegative process that represents cumulative gross investment at time  $t$ . Assume that  $(G_t)_{t \geq 0}$  is progressively measurable with respect to  $(\mathcal{F}_t^{(x, \varepsilon)})_{t \geq 0}$ . Within the present model, the net change of capital stock at time  $t$  satisfies  $dk_t = dG_t - \delta k_t dt$  and firm value can be written in each regime  $i$  as

$$V(x_t, k_t, i) \equiv \max_{\{dG_{t+u} \geq 0\}} E \left\{ \int_0^{+\infty} e^{-\rho u} [\pi(x_{t+u}, k_{t+u}) du - p dG_{t+u}] | \mathcal{F}_t^{(x, \varepsilon)} \right\}. \quad (4)$$

In this equation,  $E(\cdot | \mathcal{F}_t^{(x, \varepsilon)})$  is the expectation operator associated with the measure  $P$  conditional on the information available at time  $t$ . Moreover, since  $G_t$  is not differentiable, the last term in Eq. (4) has to be interpreted as a Stieltjes integral.

Because the demand shift  $(x_t)_{t \geq 0}$  is governed by the Markov regime switching process (2), the relevant state space is  $\{(x, \varepsilon) : x \in \mathbb{R}_{++}, \varepsilon = 1, 2\}$ . This implies that the

<sup>4</sup>The assumption of 2-state regime shifts is made here for tractability. Hamilton [17], Bansal and Zhou [5] and Guo [15] also model the regime shift process as a finite state Markov process.

optimization problem (4) is more difficult to solve than traditional models of investment (see [26]) since there can be a discontinuous jump over the investment boundary when the process  $(\varepsilon_t)_{t \geq 0}$  shifts from one state to another. This also implies that the model can generate richer investment strategies than one-regime models. In particular, we show below that the firm may increase capacity either following an increase of the demand shock in a given regime or following a regime shift.

*Solution technique:* Let  $x_i^*(k)$  be the value of the demand shock that triggers investment in regime  $i$ . Depending on parameter values, the two thresholds may be ordered differently. For expositional purpose, we analyze the case where  $x_2^*(\kappa) > x_1^*(\kappa)$  for  $\kappa \in [k, +\infty)$ . However, we present a complete characterization of the solution in Theorem 1.

Using standard techniques, it is possible to show that the Hamilton–Jacobi–Bellman equation associated with the optimization problem (4) is

$$\rho V(x_t, k_t, \varepsilon_t) = \pi(x_t, k_t) - \delta k V_k(x_t, k_t, \varepsilon_t) + \mu_{\varepsilon_t} x V_x(x_t, k_t, \varepsilon_t) + \frac{1}{2} \sigma_{\varepsilon_t}^2 x^2 V_{xx}(x_t, k_t, \varepsilon_t) + \lambda_{\varepsilon_t} [V(x_t, k_t, 3 - \varepsilon_t) - V(x_t, k_t, \varepsilon_t)], \quad (5)$$

where the Kuhn–Tucker conditions for the maximization are  $V_k(x_t, k_t, \varepsilon_t) \leq p$ ,  $dG_t \geq 0$  and  $[V_k(x_t, k_t, \varepsilon_t) - p]dG_t = 0, \forall t \geq 0$ . The left-hand side of (5) reflects the required rate of return for investing in the firm. The right-hand side is the expected change in firm value in the region for the state variable where the firm does not invest. This equation is similar to that obtained in one-regime investment models where the state variable is governed by a diffusion process.<sup>5</sup> However, it contains an additional term  $\lambda_{\varepsilon_t} [V(x_t, k_t, 3 - \varepsilon_t) - V(x_t, k_t, \varepsilon_t)]$ , that reflects the impact of the possibility of a regime shift on the value function. This term corresponds to the probability weighted change in firm value due to a regime shift.

Eq. (5) holds identically in  $k$ . Thus, the partial derivative of the left-hand side with respect to  $k$  equals the partial derivative of the right-hand side with respect to  $k$ . Performing this partial differentiation yields:

$$\rho V_k(x_t, k_t, \varepsilon_t) = \pi_k(x_t, k_t) - \delta [V_k(x_t, k_t, \varepsilon_t) + k V_{kk}(x_t, k_t, \varepsilon_t)] + \mu_{\varepsilon_t} x V_{xk}(x_t, k_t, \varepsilon_t) + \frac{\sigma_{\varepsilon_t}^2}{2} x^2 V_{xxk}(x_t, k_t, \varepsilon_t) + \lambda_{\varepsilon_t} [V_k(x_t, k_t, 3 - \varepsilon_t) - V_k(x_t, k_t, \varepsilon_t)]. \quad (6)$$

<sup>5</sup>In the diffusion case, the demand shift parameter is governed by the geometric Brownian motion  $dx_t = \mu x_t dt + \sigma x_t dW_t$ . By an application of Itô’s lemma, the expected change in firm value per unit of time is then given by

$$\frac{1}{dt} E[dV(x, k)] = -\delta k V_k(x, k) + \mu x V_x(x, k) + \frac{1}{2} \sigma^2 x^2 V_{xx}(x, k),$$

where subscripts denote partial derivatives. The equilibrium expected return on firm value is  $\rho$ . Combining this condition with the above equation gives the differential equation:

$$\rho V(x, k) = \pi(x, k) - \delta k V_k(x, k) + \mu x V_x(x, k) + \frac{1}{2} \sigma^2 x^2 V_{xx}(x, k),$$

which is solved subject to appropriate boundary conditions.

To solve Eq. (6), we will use the fact that the value function  $V$  is homogenous of degree one in  $x$  and  $k$ . This property of the value function implies that the marginal valuation of capital  $V_k$  is homogenous of degree zero in  $x_t$  and  $k_t$  and, hence, can be written simply as a function of  $y_t$ , the ratio of  $x_t$  to  $k_t$ . Define  $y_i^* = x_i^*(k)/k$ . Then  $x_2^*(k) > x_1^*(k)$  implies  $y_2^* > y_1^*$ . Define the marginal valuation of capital in regime  $i$  by:

$$q_i(y) = V_k(x, k, i). \tag{7}$$

Differentiating this equation and using the definition of  $y$  yields expressions for the partial derivatives of the value-function. Substituting the definition of  $q(y)$  and its partial derivatives in Eq. (6) yields the following system of second-order ordinary differential equations for the marginal valuation of capital  $q_i(y)$ :

- On the region  $y \leq y_1^*$ ,

$$(\rho + \delta)q_1(y) = y^\alpha + (\mu_1 + \delta)yq_1'(y) + \frac{1}{2}\sigma_1^2y^2q_1''(y) + \lambda_1[q_2(y) - q_1(y)], \tag{8}$$

$$(\rho + \delta)q_2(y) = y^\alpha + (\mu_2 + \delta)yq_2'(y) + \frac{1}{2}\sigma_2^2y^2q_2''(y) + \lambda_2[q_1(y) - q_2(y)]. \tag{9}$$

- On the region  $y_1^* \leq y \leq y_2^*$ ,

$$(\rho + \delta)q_2(y) = y^\alpha + (\mu_2 + \delta)yq_2'(y) + \frac{1}{2}\sigma_2^2y^2q_2''(y) + \lambda_2[p - q_2(y)]. \tag{10}$$

The sets  $(0, y_1^*]$  and  $[y_2^*, \infty)$  are the *inaction region* and the *action region*, respectively. We follow Guo [14] and call the set  $[y_1^*, y_2^*]$  the *transient region*. Fig. 1 illustrates these regions.

By definition of the barrier policy (see [19]), the firm makes a discrete adjustment in its capital stock, from  $k$  to  $\underline{k}$ , at the time of a shift from regime 2 to regime 1 on the region  $y_1^* \leq y \leq y_2^*$  with:<sup>6</sup>

$$\underline{k} = \sup\{\kappa \in [k, +\infty) : V_k(x, \kappa, 1) \geq p\}. \tag{11}$$

<sup>6</sup>Heuristically, one can derive the optimality of this policy following the arguments of Dixit and Pindyck [9] for impulse control of Brownian motion. Consider a small time interval  $dt$ . Since decisions are made continuously, we will be interested in the limit as  $dt \rightarrow 0$ . Suppose that the firm does not adjust capacity over the time interval  $dt$  and then increases capacity to  $\underline{k}$  at the end of this interval. The resulting value is

$$\pi(x_t, k_t, \varepsilon_t)dt + e^{-\rho dt} E^{x_t, \varepsilon_t} [V(x_{t+dt}, \underline{k}, \varepsilon_{t+dt}) - p(\underline{k} - k_t)].$$

Because the profit function  $\pi(\cdot)$  is concave in  $k$ , so is the value function  $V(\cdot)$ . This concavity property ensures that the solution to the firm's optimization problem can be found using the familiar Kuhn–Tucker conditions. The derivative of the above expression with respect to  $\underline{k}$  is

$$e^{-\rho dt} E[V_k(x_{t+dt}, \underline{k}, \varepsilon_{t+dt}) - p].$$

As  $dt \rightarrow 0$ , this expression tends to  $V_k(x_t, \underline{k}, \varepsilon_t) - p$ . Note that irreversibility requires  $\underline{k} \geq k$ . Suppose that at the time of a regime shift we have  $V_k(x_t, k, \varepsilon_t) > p$ . Since the value function is concave in  $k$ , this in turn implies that the optimal policy is to set  $\underline{k}$  at the level defined by the first-order condition  $V_k(x_t, \underline{k}, \varepsilon_t) = p$  by instantaneously installing the amount of capital  $\underline{k} - k$ .

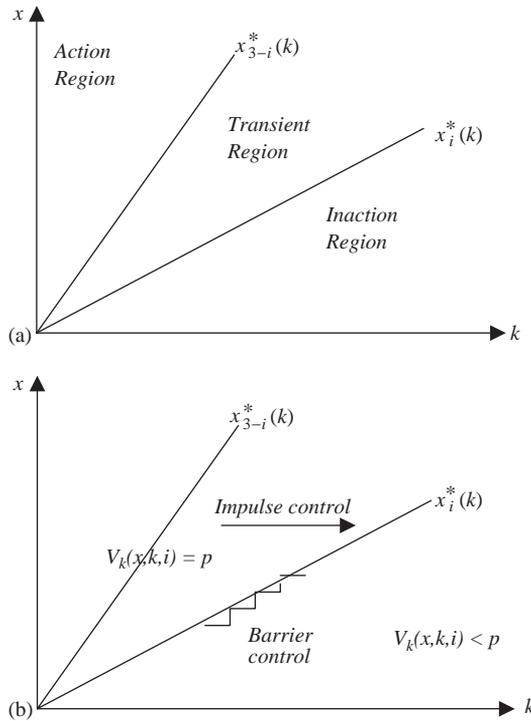


Fig. 1. Regime shifts and investment policy. (a) Represents the value-maximizing investment policy as a function of  $k$ . This investment policy requires the firm to invest in regime  $i$  if  $x_t$  exceeds  $x_i^*$ . There exists a region for the state variable  $x$  for which a shift from regime  $3 - i$  to regime  $i$  triggers investment. This region is called the transient region. (b) Represents the optimal investment policy in regime  $i$ . This investment policy is a mixture of impulse control in the transient region and barrier—or diffusion—control at  $x_i^*(k)$ .

That is, when  $y = x/k \in [y_1^*, y_2^*]$ , it is optimal for the firm to have a lumpy adjustment in capacity following a regime shift by moving the point  $(x, k)$  horizontally to the curve  $x_1^*(k)$ . Using the equality

$$V(x, k, 1) + p(\underline{k} - k) = V(x, \underline{k}, 1), \quad k < \underline{k}, \tag{12}$$

which reflects the fact that the value function is just the known value function at the terminal point of the jump minus the cost of investment, it is immediate to see that  $V_k(x, k, 1) = p$ , or  $q_1(y) = p$  on the region  $[y_1^*, y_2^*]$ . Sections 5 and 6 will provide a more detailed analysis of the nature of the optimal investment policy.

The optimization problem (4) can be solved using the set of ODEs (8)–(10) and appropriate boundary conditions. One boundary condition is given by requiring that, as the demand shift decreases, the marginal valuation of capital remains finite.

This condition can be written as

$$\lim_{y \downarrow 0} q_i(y) < \infty, \quad i = 1, 2. \tag{13}$$

Now, suppose that the firm exercises its expansion option when the state variable  $y$  reaches a trigger value  $y_i^* \equiv x_i^*/k$ . At that time,  $k$  increases by the infinitesimal increment  $dk$  and the firm pays  $pdk$ . Thus, the following condition is satisfied:  $V(x_i^*, k, i) = V(x_i^*, k + dk, i) - pdk$ ,  $i = 1, 2$ . Dividing by the increment  $dk$ , these conditions can be written in derivative form as

$$q_i(y_i^*) = p, \quad i = 1, 2. \tag{14}$$

As shown by (14), the marginal valuation of capital equals the purchase price of capital when the firm is undertaking investment. To ensure that investment occurs along the optimal path, we also require a continuity of the slopes at the endogenous investment thresholds:  $V_x(x_i^*, k, i) = V_x(x_i^*, k + dk, i)$ ,  $i = 1, 2$ . These *high-contact* conditions can be written in derivative form as (see [12]):

$$q'_i(y_i^*) = 0, \quad i = 1, 2. \tag{15}$$

Finally, because the marginal revenue product of capital  $\pi_k(\cdot)$  is a (piecewise) continuous, borel-bounded function, the marginal value-functions  $q_i(\cdot)$  are piecewise  $C^2$  (see [22, Theorem 4.9, p. 271]). Therefore, the marginal valuation of capital is  $C^0$  and  $C^1$  and satisfies the following conditions:

$$\lim_{y \downarrow y_1^*} q_2(y) = \lim_{y \uparrow y_1^*} q_2(y), \tag{16}$$

$$\lim_{y \downarrow y_1^*} q'_2(y) = \lim_{y \uparrow y_1^*} q'_2(y), \tag{17}$$

which ensure the smoothness of the marginal value function  $q_2(\cdot)$  at the boundary between the inaction region and the transient region.

#### 4. Value-maximizing investment policy

Using the set of ODEs (8)–(10) and the boundary conditions (14)–(17), it is possible to characterize the value-maximizing investment policy. Before presenting the solution to the firm’s optimization problem, we introduce the following notations. Let  $\gamma_1$  and  $\gamma_2$  be the two positive real roots of the quartic equation

$$H_1(\gamma)H_2(\gamma) = \lambda_1\lambda_2 \tag{18}$$

and let  $\beta_1$  and  $\beta_2$  be the two real roots of the quadratic equation

$$H_2(\beta) = 0 \tag{19}$$

and  $\widehat{\beta}_1$  and  $\widehat{\beta}_2$  be the two real roots of the quadratic equation

$$H_1(\beta) = 0,$$

where

$$H_i(\gamma) \equiv \rho + \delta + \lambda_i - \gamma(\mu_i + \delta) - \frac{1}{2} \sigma_i^2 \gamma(\gamma - 1). \tag{20}$$

Define  $l_i, \widehat{l}_i,$  and  $F_i, i = 1, 2,$  by

$$l_i \equiv \frac{1}{\lambda_i} \left[ \rho + \delta + \lambda_i - \gamma_i(\mu_i + \delta) - \frac{1}{2} \gamma_i(\gamma_i - 1) \sigma_i^2 \right], \tag{21}$$

$$\widehat{l}_i \equiv \frac{1}{\lambda_2} \left[ \rho + \delta + \lambda_2 - \gamma_i(\mu_2 + \delta) - \frac{1}{2} \gamma_i(\gamma_i - 1) \sigma_2^2 \right], \tag{22}$$

$$F_i \equiv \frac{\rho + \delta + \lambda_1 + \lambda_2 - \alpha(\mu_{3-i} + \delta) - \frac{1}{2} \sigma_{3-i}^2 \alpha(\alpha - 1)}{H_1(\alpha)H_2(\alpha) - \lambda_1 \lambda_2}. \tag{23}$$

The following results.

**Theorem 1.** Assume that  $H_i(\alpha) > 0, F_i > 0.$  If there exists a solution  $R \in (0, 1)$  to the nonlinear equation

$$\begin{aligned} & \frac{\frac{\beta_2 R^{\beta_1} - \beta_1 R^{\beta_2}}{\beta_2 - \beta_1} \frac{\rho + \delta}{\rho + \delta + \lambda_2} + \frac{\lambda_2}{\rho + \delta + \lambda_2} + \frac{l_2 \gamma_1 - l_1 \gamma_2}{\gamma_2 - \gamma_1}}{\frac{(\alpha - \beta_1) R^{\beta_2} - (\alpha - \beta_2) R^{\beta_1}}{(\beta_2 - \beta_1) H_2(\alpha)} + \left[ F_2 + \frac{l_1(\alpha - \gamma_2) - l_2(\alpha - \gamma_1)}{\gamma_2 - \gamma_1} F_1 - \frac{1}{H_2(\alpha)} \right] R^\alpha} \\ &= \frac{\frac{\beta_1 \beta_2 (R^{\beta_1} - R^{\beta_2})}{\beta_2 - \beta_1} \frac{\rho + \delta}{\rho + \delta + \lambda_2} + \frac{\gamma_1 \gamma_2 (l_2 - l_1)}{\gamma_2 - \gamma_1}}{\frac{\beta_2 (\alpha - \beta_1) R^{\beta_2} - \beta_1 (\alpha - \beta_2) R^{\beta_1}}{(\beta_2 - \beta_1) H_2(\alpha)} + \left[ \alpha F_2 + \frac{\gamma_1 l_1 (\alpha - \gamma_2) - \gamma_2 l_2 (\alpha - \gamma_1)}{\gamma_2 - \gamma_1} F_1 - \frac{\alpha}{H_2(\alpha)} \right] R^\alpha}, \end{aligned} \tag{24}$$

then the investment policy that maximizes firm value is characterized by the investment thresholds  $y_1^*$  and  $y_2^*$  satisfying

$$y_1^* = R y_2^* \tag{25}$$

and

$$(y_2^*)^\alpha = P \frac{\frac{\beta_2 R^{\beta_1} - \beta_1 R^{\beta_2}}{\beta_2 - \beta_1} \frac{\rho + \delta}{\rho + \delta + \lambda_2} + \frac{\lambda_2}{\rho + \delta + \lambda_2} + \frac{l_2 \gamma_1 - l_1 \gamma_2}{\gamma_2 - \gamma_1}}{\frac{(\alpha - \beta_1) R^{\beta_2} - (\alpha - \beta_2) R^{\beta_1}}{(\beta_2 - \beta_1) H_2(\alpha)} + \left[ F_2 + \frac{l_1(\alpha - \gamma_2) - l_2(\alpha - \gamma_1)}{\gamma_2 - \gamma_1} F_1 - \frac{1}{H_2(\alpha)} \right] R^\alpha}. \tag{26}$$

Otherwise, there exists a solution  $R \in (0, 1)$  to the nonlinear equation:

$$\begin{aligned} & \frac{\frac{\widehat{\beta}_2 \widehat{R}^{\widehat{\beta}_1} - \widehat{\beta}_1 \widehat{R}^{\widehat{\beta}_2}}{\widehat{\beta}_2 - \widehat{\beta}_1} \frac{\rho + \delta}{\rho + \delta + \lambda_1} + \frac{\lambda_1}{\rho + \delta + \lambda_1} + \frac{\widehat{l}_2 \gamma_1 - \widehat{l}_1 \gamma_2}{\gamma_2 - \gamma_1}}{\frac{(\alpha - \widehat{\beta}_1) \widehat{R}^{\widehat{\beta}_2} - (\alpha - \widehat{\beta}_2) \widehat{R}^{\widehat{\beta}_1}}{(\widehat{\beta}_2 - \widehat{\beta}_1) H_1(\alpha)} + \left[ F_1 + \frac{\widehat{l}_1(\alpha - \gamma_2) - \widehat{l}_2(\alpha - \gamma_1)}{\gamma_2 - \gamma_1} F_2 - \frac{1}{H_1(\alpha)} \right] R^\alpha} \\ &= \frac{\frac{\widehat{\beta}_1 \widehat{\beta}_2 (\widehat{R}^{\widehat{\beta}_1} - \widehat{R}^{\widehat{\beta}_2})}{\widehat{\beta}_2 - \widehat{\beta}_1} \frac{\rho + \delta}{\rho + \delta + \lambda_1} + \frac{\gamma_1 \gamma_2 (\widehat{l}_2 - \widehat{l}_1)}{\gamma_2 - \gamma_1}}{\frac{\widehat{\beta}_2 (\alpha - \widehat{\beta}_1) \widehat{R}^{\widehat{\beta}_2} - \widehat{\beta}_1 (\alpha - \widehat{\beta}_2) \widehat{R}^{\widehat{\beta}_1}}{(\widehat{\beta}_2 - \widehat{\beta}_1) H_1(\alpha)} + \left[ \alpha F_1 + \frac{\gamma_1 \widehat{l}_1 (\alpha - \gamma_2) - \gamma_2 \widehat{l}_2 (\alpha - \gamma_1)}{\gamma_2 - \gamma_1} F_2 - \frac{\alpha}{H_1(\alpha)} \right] R^\alpha}, \end{aligned} \tag{27}$$

and the optimal investment policy is characterized by the thresholds  $y_1^*$  and  $y_2^*$  defined by

$$y_2^* = Ry_1^* \tag{28}$$

and

$$(y_1^*)^\alpha = p \frac{\frac{\widehat{\beta}_2 R^{\beta_1} - \widehat{\beta}_1 R^{\beta_2}}{\widehat{\beta}_2 - \widehat{\beta}_1} \frac{\rho + \delta}{\rho + \delta + \lambda_1} + \frac{\lambda_1}{\rho + \delta + \lambda_1} + \frac{\widehat{l}_2 \gamma_1 - \widehat{l}_1 \gamma_2}{\gamma_2 - \gamma_1}}{\frac{(\alpha - \widehat{\beta}_1) R^{\beta_2} - (\alpha - \widehat{\beta}_2) R^{\beta_1}}{(\widehat{\beta}_2 - \widehat{\beta}_1) H_1(\alpha)} + \left[ F_1 + \frac{\widehat{l}_1(\alpha - \gamma_2) - \widehat{l}_2(\alpha - \gamma_1)}{\gamma_2 - \gamma_1} F_2 - \frac{1}{H_1(\alpha)} \right] R^\alpha}.$$

**Proof.** See Appendix A.  $\square$

**Remark.** Eq. (18) must have two positive real roots. To see this, define

$$h(\gamma) = H_1(\gamma)H_2(\gamma) - \lambda_1\lambda_2. \tag{29}$$

Then,  $h$  is continuous and satisfies  $h(0) > 0$ ,  $h(-\infty) > 0$ ,  $h(\infty) > 0$ , and  $h(\beta_i) = -\lambda_1\lambda_2 < 0$ ,  $i = 1, 2$ . Since  $\beta_1\beta_2 = -2(\rho + \delta + \lambda_2)/\sigma_2^2 < 0$ , it follows that the equation  $h(\gamma) = 0$  has two positive roots by the intermediate value theorem.

### 5. Discussion and simulations

Theorem 1 characterizes the investment policy that maximizes firm value when the dynamics of the demand shift are governed by (2). This investment policy takes the form of a trigger policy and there exists one trigger threshold for each regime. Since the two regimes are related to one another (through the persistence parameters  $\lambda_i$ ), the trigger function in each regime has to reflect the possibility for the firm to adjust capacity in the other regime. In other words, the firm has to determine an exercise strategy for its options to adjust capacity in each regime, while taking into account the optimal investment strategy in the other regime.

Theorem 1 demonstrates that when determining the timing of investment, the firm balances the marginal increase in expected cash flows with purchase price of assets. This trade-off shows up for example in Eq. (26) which can be written as (remembering by (1) that  $\pi_k = y^\alpha$ ):

$$F_2\pi_k(x_2^*, k_t) = p\Theta \tag{30}$$

in which  $x_2^* \equiv y_2^*/k$ ,  $F_2$  is defined in Eq. (23), and  $\Theta$  is a positive constant. The left-hand side of Eq. (30) accounts for the change in the expected present value of the firm cash flows associated with a marginal increase of capacity in regime 2 when  $x = x_2^*$ . That is, using the Feynman Kac theorem, it is possible to show that the following equality holds:

$$F_i\pi_k(x_t, k_t) = E^{x_t, i} \left\{ \int_0^{+\infty} e^{-(\rho + \delta)u} \pi_k(x_{t+u}, k_{t+u}) du \right\}, \tag{31}$$

where  $E^{x,i}(\cdot)$  is the expectation operator associated with the measure  $P$  conditional on  $x_t = x$  and  $\varepsilon_t = i$ . The right-hand side of (30) is the cost associated with an increase in capacity. As usual for investment decisions under uncertainty, this cost has two components: The purchase price of capital  $p$  and the value of waiting to invest represented by  $\Theta$ . Thus, for a given productivity of assets, the investment threshold should decrease with those very parameters that increase  $\Theta$ . At the same time, the decision to invest should be hastened by smaller costs of exercising the option. This second effect is directly illustrated by Theorem 1 that shows that the thresholds  $y_i^*$ ,  $i = 1, 2$ , are linearly increasing in the purchase price of capital  $p$ .

An analogy with option pricing theory tells us that input parameters such as the drift, volatility, and the persistence in each regime should enter the decision to invest. Consider first the impact of the persistence in regimes on the optimal investment strategy. Because the persistence in regimes reflects the opportunity cost of investing in one regime vs. the other, the ratio of the two investment thresholds is affected by its changes. Specifically, a lower persistence of regime  $i$  (i.e. a higher  $\lambda_i$ ) reduces the opportunity cost of investing in regime  $i$ , and hence narrows the gap between the investment thresholds in the two regimes.

Consider next the impact of the drift and volatility parameters. Traditional investment models show that the option of waiting to invest has more value when uncertainty is greater (see [24] or [26]). This implies that in each regime  $i$  the investment threshold  $y_i^*$  increases with  $\sigma_i$ . This also implies that the ratio  $y_2^*/y_1^*$  (resp.  $y_1^*/y_2^*$ ) increases with the volatility of regime 2 (resp. regime 1) and decreases with the volatility of regime 1 (resp. regime 2). Additionally, the ratio  $y_2^*/y_1^*$  decreases with the drift of the gain process in regime 1 and increases with the drift of the gain process in regime 2. (This effect essentially arises because of the impact of the drift parameter on  $F_i$ .) Finally, two additional results are worth being mentioned. First, the impact of the drift and volatility parameters on the value-maximizing investment thresholds is not as important as in traditional real options model because of the possibility of a regime shift. Second, whenever  $\lambda_i \neq 0$ , changes in the dynamics of the demand shock in regime  $i$  affect not only the investment threshold in that regime ( $y_i^*$ ) but also the investment threshold in the other regime ( $y_{3-i}^*$ ).

Table 1 provides a number of simulation results relating the ratio  $R = y_1^*/y_2^*$  of the two trigger thresholds to input parameter values. The base case environment is as follows: The risk-free interest rate  $\rho = 6\%$ , the depreciation rate  $\delta = 0$ , the drift and volatility parameters in the first regime  $\mu_1 = 0.04$  and  $\sigma_1 = 0.2$ , the drift and volatility parameters in the second regime  $\mu_2 = 0.01$  and  $\sigma_2 = 0.3$ , the persistence of the gain process  $\lambda_1 = 0.15$  and  $\lambda_2 = 0.1$ , and the productivity of assets in place  $1 - \alpha = 0.47$ .<sup>7</sup> Results in Table 1 are consistent with the above discussion. They also

<sup>7</sup>The profit function described by Eq. (1) can be approximated as follows. Let the firm production be Cobb–Douglas and homogeneous of degree one with respect to capital and labor, with capital share  $1 - \phi$ . Let the demand faced by the firm be isoelastic, with price elasticity  $1/(\theta - 1)$ . It follows from this specification that the share of profits going to capital depends on  $\phi$  and  $\theta$  through the following relation:  $1 - \alpha = (1 - \theta)\phi/(1 - \theta\phi)$ . Labor's share of national income in U.S. postwar data has been relatively constant over time at  $\phi = 0.64$  despite the increase in real wages (see [23]). If  $\theta = 0.5$ , we have  $1 - \alpha \cong 0.47$ .

Table 1  
Simulation results

$\rho$	4%	5%	6%	7%	8%	9%
$R$	0.7612	0.7704	0.7784	0.7856	0.7920	0.7978
$\mu_1$	0.02	0.03	0.04	0.05	0.06	0.07
$R$	0.8064	0.7917	0.7784	0.7664	0.7557	0.7462
$\mu_2$	0.04	0.08	0.12	0.16	0.2	0.24
$R$	0.8236	0.8830	0.9373	0.9834	1.0177	1.0434
$\sigma_1$	0.1	0.2	0.3	0.4	0.5	0.6
$R$	0.6596	0.7784	0.9427	1.1675	1.4557	1.8153
$\sigma_2$	0.1	0.2	0.3	0.4	0.5	0.6
$R$	1.1545	0.9447	0.7784	0.6397	0.5249	0.4307
$\lambda_1$	0.05	0.1	0.15	0.2	0.25	0.3
$R$	0.7309	0.7581	0.7784	0.7944	0.8073	0.8181
$\lambda_2$	0.05	0.1	0.15	0.2	0.25	0.3
$R$	0.7658	0.7784	0.7890	0.7982	0.8062	0.8132

The Table provides comparative statics regarding the ratio of the investment thresholds  $R = y_1^*/y_2^*$ . Input parameter values are set as follows: The risk-free interest rate  $r = 6\%$  the depreciation rate  $\delta = 0$ , the drift and volatility parameters in regime 1:  $\mu_1 = 0.04$  and  $\sigma_1 = 0.2$ , the drift and volatility parameters in regime 2:  $\mu_2 = 0.01$  and  $\sigma_2 = 0.3$ , the persistence of the demand shock  $\lambda_1 = 0.15$  and  $\lambda_2 = 0.1$ , and the productivity of assets in place  $\alpha = 0.53$ . The effects of changing these parameters are also examined.

reveal that depending on parameter values, the two investment thresholds may switch orders.

Some of the effects discussed above are also depicted in Fig. 2 which plots the ratio  $R^{-1} = y_2^*/y_1^*$  as a function of the volatility and persistence of the gain process in the base case environment (where input parameters are such that  $y_2^* > y_1^*$ ).

Consistent with the above arguments, Fig. 2 reveals that for a given degree of persistence in regimes, a higher volatility of the demand shift parameter in regime 2 increases the gap between the investment thresholds in the two regimes. For a given volatility of the demand shift parameter, a decrease in the degree of persistence in regimes (i.e. an increase in  $\lambda_i$ ) narrows the gap between the two thresholds.

## 6. Implications for capital accumulation

Theorem 1 characterizes the investment policy that maximizes firm value when the dynamics of the demand shock shift between two states at random times. While this investment policy takes the form of a trigger policy as in traditional one-regime

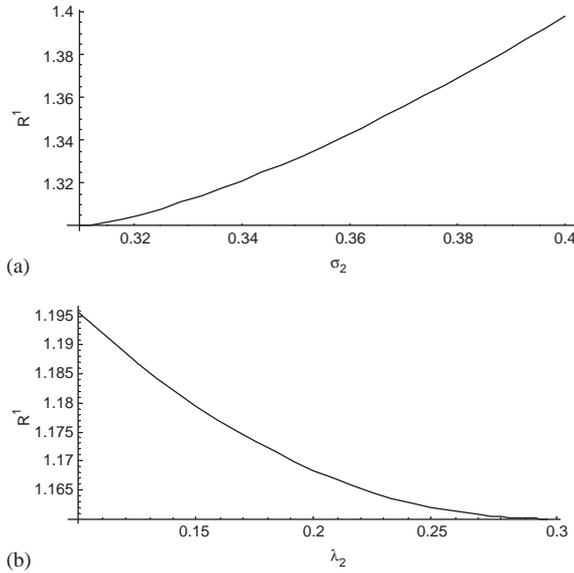


Fig. 2. Ratio of the investment thresholds. The figure plots the ratio  $R = y_1^*/y_2^*$  relating the investment thresholds in the two regimes as a function of the volatility and persistence of the marginal revenue product of capital. Input parameter values for the base case environment are set as follows: The risk-free interest rate  $r = 6\%$ , the depreciation rate  $\delta = 0$ , the drift and volatility parameters in regime 1:  $\mu_1 = 0.04$  and  $\sigma_1 = 0.2$ , the drift and volatility parameters in regime 2:  $\mu_2 = 0.01$  and  $\sigma_2 = 0.3$ , the persistence of the demand shock  $\lambda_1 = 0.15$  and  $\lambda_2 = 0.1$ , and the productivity of assets in place  $\alpha = 0.53$ . In this environment we have  $y_2^* > y_1^*$  and  $R^{-1} > 1$ . (a) Investment threshold ratio  $R^{-1}$  as a function of  $\sigma_2$ ; (b) investment threshold ratio  $R^{-1}$  as a function of  $\lambda_2$ .

models, two major differences arise within the present model. First, this optimal investment policy is characterized by a different trigger threshold  $x_i^*(k)$  for each regime  $i$ . This implies that, while there exists a region of the state space, the *action region*, where it is optimal to invest independently of the current regime ( $\mathcal{A} = \{(x, \varepsilon) \in \mathbb{R}_{++} \times \{1, 2\} : V_k(x, k, \varepsilon) \geq p\}$ ), there exists another region, the *transient region*, where it is only optimal to invest if  $\varepsilon$  is in state 1 ( $\mathcal{R} = \{(x, \varepsilon) \in \mathbb{R}_{++} \times \{1, 2\} : V_k(x, k, 1) = p, V_k(x, k, 2) < p\}$ ). Second, because of the possibility of a regime shift, the optimal trigger threshold in each regime reflects the possibility for the firm to invest in the other regime. In particular, the transient region  $\mathcal{R}$  is non empty whenever  $\lambda_i \neq 0, i = 1, 2$ .

An important question is whether regime shifts actually affect growth and capital accumulation. To answer this question, it is necessary to examine the implications of the value-maximizing investment policy for the rate of investment. One important prediction of the present model is that discrete adjustments in the firm’s capital stock can occur several times throughout the lifetime of the firm, even though there are no fixed adjustment costs in the model. This is in contrast with traditional models of capacity adjustment in which investment is infinitesimal in the absence of fixed costs and such a discrete adjustment may occur only at the initial

instant if the state of the system is above the optimal investment curve  $x_i^*(k)$ .<sup>8</sup> For instance, Dixit and Pindyck [9, p. 362] note: “At the initial instant the point [of the system] may be above the [investment] curve either because nonoptimal policies were followed in the past or because some unexpected shock just moved the curve”.

The analysis in Sections 3 and 4 shows that when the demand shock follows the Markov regime switching model (2), such “unexpected shocks” may arise at random times in the future, inducing lumpy adjustments in capacity. In other words, the optimal investment behavior of the firm can be potentially characterized by three regimes. When the demand shock is below the investment curve  $x_i^*(k)$  in regime  $i$ , the optimal rate of investment is zero. When the demand shock reaches this curve from below, it is optimal to adjust marginally the capital stock. When the demand shock reaches the lowest of the two investment curves from above, the adjustment of capacity follows a regime shift and is discrete. This investment policy is consistent with the evidence reported by Abel and Eberly [1]. Using panel data to estimate a model of optimal investment, they find that there is a temporal concentration of investment—investment is intermittent—and that the rate of investment typically is very small but occasionally exhibits some spurts of growth.

Interestingly, there is an asymmetry between the highest investment curve and the lowest one. In fact, because we presume that investment is irreversible, a shift in capital only occurs if the situation brightens up (a shift from the regime with the highest curve to the regime with the lowest curve). In addition, the size of the jump in capacity following a regime shift is not constant but depends on the value of the demand shock at the time of the regime shift as well as current firm size. This property of the optimal investment policy provides an important step towards the “realistic and empirically important feature that units do not always wait for the same stock disequilibrium to adjust, and that adjustments are not always the same size across firms and for the same firm over time [...]” [7]. To see this, assume that parameter values are such that  $H_i(\alpha) > 0$ ,  $F_i > 0$ , and  $x_2^* > x_1^*$ . In addition, suppose that the state is currently in regime 2,  $\varepsilon(t) = 2$ , and belongs to the transient region,  $x_t \in [x_1^*(k), x_2^*(k)]$ . Then, a shift in regimes induces a discrete adjustment in firm size from  $k$  to  $\underline{k}$ , in which  $\underline{k}$  satisfies:

$$\underline{k} = \{\kappa \in [k, \infty) : x_1^*(\kappa) = x_t\}. \quad (32)$$

Eq. (32) shows that the jump in capacity following a regime shift in the transient region is given by  $\underline{k} - k$ , where  $\underline{k}$  is the capital stock on the curve  $x_1^*(\kappa)$  such that the point  $(x_t, k)$  moves horizontally to this curve. This feature of the optimal investment policy can be better understood by reformulating the optimal policy in terms of marginal revenue product of capital (see Section 7). Optimality requires the firm to adjust its capital stock as needed to maintain the marginal revenue product of capital

<sup>8</sup> Models that include fixed costs of adjustments also have jumps in the capital stock. See for example [4] or [7]. The former paper generalizes the results in Hayashi [20] to the stochastic case and characterizes the optimal investment policy in the presence of adjustment costs. The latter develops an  $(S, s)$  model of investment in which adjustment costs are time-varying and capacity adjustments are lumpy and differ in size.

$y^\alpha = (x_t/k)^\alpha$  equal to  $(y_1^*)^\alpha$  given in Theorem 1. Using (32), we see that optimality then implies an increase in capacity from  $k$  to  $\underline{k}$  defined by

$$\frac{x_t}{\underline{k}} = \frac{x_1^*(\underline{k})}{\underline{k}} = y_1^*$$

or

$$\underline{k} = \frac{x_t}{R} \left[ \frac{\frac{(\alpha-\beta_1)R^{\beta_2} - (\alpha-\beta_2)R^{\beta_1}}{(\beta_2-\beta_1)H_2(\alpha)} + \left( F_2 + \frac{l_1(\alpha-\gamma_2) - l_2(\alpha-\gamma_1)}{\gamma_2-\gamma_1} F_1 - \frac{1}{H_2(\alpha)} \right) R^\alpha}{p \left( \frac{\beta_2 R^{\beta_1} - \beta_1 R^{\beta_2}}{\beta_2 - \beta_1} \frac{\rho + \delta}{\rho + \delta + \lambda_2} + \frac{\lambda_2}{\rho + \delta + \lambda_2} + \frac{l_2 \gamma_1 - l_1 \gamma_2}{\gamma_2 - \gamma_1} \right)} R^\alpha \right]^{\frac{1}{\alpha}} \tag{33}$$

Thus, an additional implication of the model is that the size of the jump following a regime shift is increasing with the contemporaneous value of the demand shock  $x$ . Since marginal  $q$  also increases with  $x$ , this in turn implies that the size of the jump increases with marginal  $q$ .

### 7. Marginal Q and the user cost of capital

The analysis so far has focused on the characterization of the dynamic behavior of investment when changes in the demand shift parameter are governed by (2). Another question of interest relates to the determinants of marginal  $q$  and the user cost of capital in such an environment.

*Marginal q:* Section 4 characterizes the investment policy that maximizes firm value. This investment policy relies on the boundary conditions (14) and (15) that ensure the smoothness of the marginal value functions at the selected trigger levels. In addition to providing a solution to the firm’s problem, the system of ODEs (14) and (15) allows for a determination of the marginal valuation of capital, i.e. scaled marginal  $q$ . In particular, we have the following result.

**Theorem 2.** Assume that  $H_i(\alpha) > 0$ ,  $F_i > 0$ , and  $y_2^* > y_1^*$  where the trigger levels  $y_1^*$  and  $y_2^*$  are defined in Theorem 1. Then, the marginal valuation of capital in regime  $i$  is given by

$$V_k(x, k, i) = q_i(y), \quad i = 1, 2, \tag{34}$$

where

$$q_1(y) = \begin{cases} A_1 y^{\gamma_1} + A_2 y^{\gamma_2} + F_1 y^\alpha, & y \leq y_1^*, \\ p, & y \geq y_1^*, \end{cases} \tag{35}$$

and

$$q_2(y) = \begin{cases} l_1 A_1 y^{\gamma_1} + l_2 A_2 y^{\gamma_2} + F_2 y^\alpha, & y \leq y_1^*, \\ C_1 y^{\beta_1} + C_2 y^{\beta_2} + \frac{y^\alpha}{H_2(\alpha)} + \frac{\lambda_2 p}{\rho + \delta + \lambda_2}, & y_1^* \leq y \leq y_2^*, \\ p, & y \geq y_2^*. \end{cases} \tag{36}$$

In Eqs. (35) and (36) the factors  $H_i$ ,  $\gamma_i$ ,  $\beta_i$  and  $l_i$  are determined by Eqs. (18)–(20). In addition, the factors  $A_1$ ,  $A_2$ ,  $C_1$  and  $C_2$  are defined by

$$A_1 = \frac{1}{(\gamma_2 - \gamma_1)(y_1^*)^{\gamma_1}} [\gamma_2 p + (\alpha - \gamma_2) F_1(y_1^*)^\alpha], \quad (37)$$

$$A_2 = \frac{1}{(\gamma_1 - \gamma_2)(y_1^*)^{\gamma_2}} [\gamma_1 p + (\alpha - \gamma_1) F_1(y_1^*)^\alpha] \quad (38)$$

and

$$C_1 = \frac{1}{(\beta_2 - \beta_1)(y_2^*)^{\beta_1}} \left[ \frac{(\alpha - \beta_2)(y_2^*)^\alpha}{H_2(\alpha)} + \frac{\beta_2(\rho + \delta)p}{\rho + \delta + \lambda_2} \right], \quad (39)$$

$$C_2 = \frac{1}{(\beta_1 - \beta_2)(y_2^*)^{\beta_2}} \left[ \frac{(\alpha - \beta_1)(y_2^*)^\alpha}{H_2(\alpha)} + \frac{\beta_1(\rho + \delta)p}{\rho + \delta + \lambda_2} \right]. \quad (40)$$

*Discussion:* Theorem 2 provides a characterization of the marginal valuation of capital when the dynamics of the gain process are governed by the Markov regime switching model (2) and the firm follows the investment policy derived in Theorem 1. These expressions generalize those that characterize marginal  $q$  in the diffusion case to incorporate possible regime shifts in the dynamics of operating profits. Indeed, it is immediate to see that when  $\lambda_1 = \lambda_2 = 0$ , Eqs. (35) and (36) simplify to (see for example the value of marginal  $q$  in the transient region):

$$q(y) = A \left( \frac{y}{y^*} \right)^\beta + \frac{y^\alpha}{\rho + \delta - \alpha(\mu + \delta) - \frac{1}{2} \sigma^2 \alpha(\alpha - 1)} \quad (41)$$

in which

$$A = \frac{\alpha}{\alpha - \beta} p < 0$$

and  $y^*$  is the value-maximizing investment threshold defined by

$$(y^*)^\alpha = \frac{\beta}{\beta - \alpha} \left[ \rho + \delta - \alpha(\mu + \delta) - \frac{1}{2} \sigma^2 \alpha(\alpha - 1) \right] p, \quad (42)$$

where  $\beta$  is the positive root of the quadratic equation

$$\rho + \delta - (\mu + \delta)\beta - \frac{1}{2} \sigma^2 \beta(\beta - 1) = 0. \quad (43)$$

Eq. (41) shows that in traditional models of capacity choice,<sup>9</sup> the marginal valuation of capital has two components: The contribution of this marginal unit to the profit flow (second term on the right-hand side) and the marginal option value to adjust capacity (first term on the right-hand side). Note that this second component is negative since when the firm invests in a marginal unit of capital it gives up the valuable option of waiting to invest in this unit.

<sup>9</sup> See for example [26,3,6].

Theorem 2 reveals that the expression for marginal  $q$  is more complex when changes in demand shift parameter are governed by Eq. (2). In the inaction region for example, marginal  $q$  has three components. First, it incorporates the expected present value of the profits generated by the marginal unit of capital. Second, it reflects the change in marginal  $q$  due to a potential regime shift. Third, it captures the change in marginal  $q$  arising if and when the decision variable reaches the investment boundary  $y_i^*$ .

In the transient region  $[y_1^*, y_2^*]$ , the marginal valuation of capital has four components and can be written as

$$q_2(y) = \underbrace{C_1 y^{\beta_1}}_{(1)} + \underbrace{C_2 y^{\beta_2}}_{(2)} + \underbrace{y^\alpha / H_2(\alpha)}_{(3)} + \underbrace{\lambda_2 p / (\rho + \delta + \lambda_2)}_{(4)}.$$

The first term on the right-hand side of this equation accounts for the change in value arising if and when the demand shift parameter reaches the investment boundary  $y_2^*$ . The second term represents the change in value arising if and when the demand shift parameter reaches the lower boundary of the transient region. The third term is the expected present value of the profits generated by the marginal unit of capital. The fourth term reflects the probability weighted change in value arising from a regime shift.

Finally, it is interesting to note that when the state variable reaches the lower boundary of the *transient region*, there is a single exogenous change in the marginal valuation of capital. This change in value arises from the fact that a switch in the process  $\varepsilon(t)$  does not trigger the investment in the inaction region whereas it triggers investment in the transient region. When the value of the state variable reaches the *action region*, this exogenous change is accompanied by an endogenous change. In that case, the firm exercises its option to adjust capacity marginally and the value-maximizing policy is determined in Theorem 1.

*User cost of capital:* In standard neoclassical models, the capital stock is adjusted continuously so as to maintain the marginal revenue product of capital equal to the user cost of capital. With irreversibility and uncertainty, there exists a user cost of capital  $c^*$  for purchasing capital and the optimal policy is to purchase capital as needed to prevent the marginal revenue product of capital from rising above  $c^*$ . Within the present model, the user cost of capital is defined by

$$c_i \equiv (\rho + \delta)q_i(y) - \frac{1}{dt} E^{y,i}(dq_i(y)), \quad i = 1, 2. \tag{44}$$

As noted by Abel and Eberly [3], with uncertainty and irreversibility it is not the purchase price of capital which is relevant for the user cost of capital but rather its shadow price,  $q(y)$ . Moreover, as shown by Eq. (14), these two prices differ unless the firm is actually purchasing capital<sup>10</sup>. Applying Itô's lemma to  $q_i(y)$  yields:

$$c_i = (\rho + \delta)q_i(y) - (\mu_i + \delta)yq'_i(y) - \frac{1}{2}\sigma_i^2 y_i^2 q''_i(y) - \lambda_i[q_{3-i}(y) - q_i(y)]. \tag{45}$$

<sup>10</sup>In the analysis of Jorgenson [21], investment is costlessly reversible and marginal  $q$  is always equal to the purchase/sale price of capital.

Plugging this expression for the user cost of capital in the system of ODEs yield

$$\tilde{c}_i = y^\alpha, \quad i = 1, 2. \quad (46)$$

Eq. (46) demonstrates that, for a given value of the demand shift parameter in the inaction region, the user cost of capital does not depend on the current regime. By contrast, the potential range of values for the user cost of capital is regime dependent. In particular, the user cost of capital relevant for purchasing capital is  $c_1^* = (y_1^*)^\alpha$  in regime 1 whereas it is  $c_2^* = (y_2^*)^\alpha > c_1^*$  in regime 2. This feature of the model in turn implies that the marginal revenue product of capital belongs to  $(0, c_i^*]$  in regime  $i$ . In other words, the set of values for the marginal revenue product of capital in regime 1 is strictly included in the set of values for the marginal revenue product of capital in regime 2. In addition, for  $y \in [y_1^*, y_2^*]$  the ratio of the marginal revenue product of capital in regime 2 to the marginal revenue product of capital in regime 1 deviates from 1 and increases with  $y$  until the point  $y = y_2^*$  where it reaches a maximum of  $R^{-\alpha}$  with  $R \equiv y_1^*/y_2^*$ . Because the ratio of the two investment thresholds depend on the drift and volatility parameters of the two regimes and the persistence in regimes, so does the ratio of the two marginal revenue product of capital in the transient region. This result again emphasizes the regime-dependent nature of the optimal policy.

## 8. Conclusion

This paper has analyzed investment decisions under uncertainty when the dynamics of the decision variable—growth rate and diffusion coefficient—shift between different states at random times. The main analytical result of the paper is that the value-maximizing investment policy is such that in each regime the firm's investment policy is optimal, conditional on the optimal investment policy in the other regimes. This optimal investment policy is characterized by a different investment curve for each regime. Moreover, because of the possibility of a regime shift, the investment curve in each regime reflects the possibility for the firm to invest in the other regime.

To determine the implications of the model for investment decisions and capital accumulation, we showed that the state space of the dynamic investment problem can be partitioned into various domains including an *inaction region* where no investment occurs. Outside of this region, the optimal rate of investment can be in one of two regimes: infinitesimal or lumpy. Investment is infinitesimal following an increase of the firm cash flows in a given regime. Investment is lumpy following a shift from the regime with the highest investment curve to the regime with the lowest one. That is, the model predicts that with irreversibility and regime shifts investment is intermittent and increases with marginal  $q$ . Moreover, the optimal rate of investment typically is very small but occasionally exhibits some spurts of capacity expansion. These predictions are generally consistent with the available empirical evidence on firms' investment behavior (see [7] or [1]). The paper also provided an

analysis of the determinants of marginal  $q$  and the user cost of capital in such an environment.

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**Appendix A. Proof of Theorem 1**

We present the case where  $y_2^* > y_1^*$ . The other case follows from a symmetric argument. The general solutions to Eqs. (8) and (9) are

$$q_i(y) = \sum_{j=1}^4 A_{i,j}y^{j_i} + F_iy^\alpha, \quad i = 1, 2. \tag{A.1}$$

We set  $\gamma_j > 0$  for  $j = 1, 2$  and  $\gamma_j < 0$  for  $j = 3, 4$  by convention. The no-bubbles condition (13) implies that  $A_{i,j} = 0$  for  $j = 3, 4$  which in turn implies that Eq. (A.1) reduces to

$$q_i(y) = A_{i,1}y^{\gamma_1} + A_{i,2}y^{\gamma_2} + F_iy^\alpha, \quad i = 1, 2.$$

Substituting this solution into (8) and (9) and matching coefficients yield the expressions for  $F_i$  and  $\gamma_j$  in (23) and (18), and  $A_{2,j} = l_j A_{1,j}$  with

$$l_j = \frac{1}{\lambda_1} \left[ \rho + \delta + \lambda_1 - \gamma_j(\mu_1 + \delta) - \frac{1}{2}\gamma_j(\gamma_j - 1)\sigma_1^2 \right].$$

Solving Eq. (10) yields: For  $y \in (y_1^*(k), y_2^*(k))$ ,

$$q_2(y) = C_1y^{\beta_1} + C_2y^{\beta_2} + \frac{y^\alpha}{H_2(\alpha)} + \frac{\lambda_2 p}{\rho + \delta + \lambda_2},$$

where  $\beta_1$  and  $\beta_2$  are the two roots of the quadratic equation  $H_2(\beta) = 0$ .

The above set of equations shows that we have 6 unknowns:  $A_{1,1}$ ,  $A_{1,2}$ ,  $C_1$ ,  $C_2$ ,  $y_1^*$ , and  $y_2^*$ . We thus need 6 equations to identify these variables. They are given by the boundary conditions (14)–(17). Plugging the above functions  $q_i(y)$  in the boundary conditions (14) and (15) yields

$$A_{1,1}(y_1^*)^{\gamma_1} + A_{1,2}(y_1^*)^{\gamma_2} + F_1(y_1^*)^\alpha = p, \tag{A.2}$$

$$\gamma_1 A_{1,1}(y_1^*)^{\gamma_1-1} + \gamma_2 A_{1,2}(y_1^*)^{\gamma_2-1} + \alpha F_1(y_1^*)^{\alpha-1} = 0, \tag{A.3}$$

$$C_1(y_2^*)^{\beta_1} + C_2(y_2^*)^{\beta_2} + \frac{(y_2^*)^\alpha}{H_2(\alpha)} + \frac{\lambda_2 p}{\rho + \delta + \lambda_2} = p \quad (\text{A.4})$$

$$\beta_1 C_1(y_2^*)^{\beta_1-1} + \beta_2 C_2(y_2^*)^{\beta_2-1} + \frac{\alpha(y_2^*)^{\alpha-1}}{H_2(\alpha)} = 0. \quad (\text{A.5})$$

The solution to this set of equations is

$$A_{1,1} = \frac{1}{(\gamma_2 - \gamma_1)(y_1^*)^{\gamma_1}} [\gamma_2 p + (\alpha - \gamma_2) F_1(y_1^*)^\alpha], \quad (\text{A.6})$$

$$A_{1,2} = \frac{1}{(\gamma_1 - \gamma_2)(y_1^*)^{\gamma_2}} [\gamma_1 p + (\alpha - \gamma_1) F_1(y_1^*)^\alpha], \quad (\text{A.7})$$

$$C_1 = \frac{1}{(\beta_2 - \beta_1)(y_2^*)^{\beta_1}} \left[ \frac{(\alpha - \beta_2)(y_2^*)^\alpha}{H_2(\alpha)} + \frac{\beta_2(\rho + \delta)p}{\rho + \delta + \lambda_2} \right], \quad (\text{A.8})$$

$$C_2 = \frac{1}{(\beta_1 - \beta_2)(y_2^*)^{\beta_2}} \left[ \frac{(\alpha - \beta_1)(y_2^*)^\alpha}{H_2(\alpha)} + \frac{\beta_1(\rho + \delta)p}{\rho + \delta + \lambda_2} \right]. \quad (\text{A.9})$$

Substituting these expressions in conditions (16) and (17) yields with  $Ry_2^* = y_1^*$ :

$$(y_2^*)^\alpha = p \frac{\frac{\beta_2 R^{\beta_1} - \beta_1 R^{\beta_2}}{\beta_2 - \beta_1} \frac{\rho + \delta}{\rho + \delta + \lambda_2} + \frac{\lambda_2}{\rho + \delta + \lambda_2} + \frac{l_2 \gamma_1 - l_1 \gamma_2}{\gamma_2 - \gamma_1}}{F_2 R^\alpha + \frac{(\alpha - \beta_1) R^{\beta_2} - (\alpha - \beta_2) R^{\beta_1}}{(\beta_2 - \beta_1) H_2(\alpha)} + \frac{l_1(\alpha - \gamma_2) - l_2(\alpha - \gamma_1)}{\gamma_2 - \gamma_1} F_1 R^\alpha - \frac{R^\alpha}{H_2(\alpha)}} \quad (\text{A.10})$$

and

$$(y_2^*)^\alpha = p \frac{\frac{\beta_1 \beta_2 (R^{\beta_1} - R^{\beta_2})}{\beta_2 - \beta_1} \frac{\rho + \delta}{\rho + \delta + \lambda_2} + \frac{\gamma_1 \gamma_2 (l_2 - l_1)}{\gamma_2 - \gamma_1}}{\alpha F_2 R^\alpha + \frac{\beta_2(\alpha - \beta_1) R^{\beta_2} - \beta_1(\alpha - \beta_2) R^{\beta_1}}{(\beta_2 - \beta_1) H_2(\alpha)} + \frac{\gamma_1 l_1 (\alpha - \gamma_2) - \gamma_2 l_2 (\alpha - \gamma_1)}{\gamma_2 - \gamma_1} F_1 R^\alpha - \frac{\alpha R^\alpha}{H_2(\alpha)}}. \quad (\text{A.11})$$

Combining (A.10) with (A.11) yields Eq. (24).

## References

- [1] A. Abel, J. Eberly, Investment and  $q$  with fixed costs: an empirical analysis, unpublished manuscript, University of Pennsylvania, 2001.
- [2] A. Abel, J. Eberly, The effects of irreversibility and uncertainty on capital accumulation, *J. Mont. Econ.* 44 (1999) 339–377.
- [3] A. Abel, J. Eberly, Optimal investment with costly reversibility, *Rev. Econ. Stud.* 63 (1996) 581–593.
- [4] A. Abel, J. Eberly, A unified model of investment under uncertainty, *Amer. Econ. Rev.* 84 (1994) 1369–1384.
- [5] R. Bansal, H. Zhou, Term structure of interest rates with regime shifts, *J. Finan.* 57 (2002) 1997–2043.
- [6] G. Bertola, R. Caballero, Irreversibility and aggregate investment, *Rev. Econ. Stud.* 61 (1994) 223–246.
- [7] R. Caballero, E. Engel, Explaining investment dynamics in U.S. manufacturing: a generalized  $(S, s)$  approach, *Econometrica* 67 (1999) 783–826.

- [8] R. Chirinko, Business fixed investment spending: modeling strategies, empirical results, and policy implications, *J. Econ. Lit.* 31 (1993) 1875–1911.
- [9] A. Dixit, R. Pindyck, *Investment Under Uncertainty*, Princeton University Press, Princeton, NJ, 1994.
- [10] J. Driffill, M. Sola, Option pricing and regime switching, unpublished manuscript, University of London, 2000.
- [11] J. Driffill, M. Sola, Irreversible investment and changes in regime, unpublished manuscript, University of London, 2001.
- [12] B. Dumas, Super contact and related optimality conditions, *J. Econ. Dynam. Control* 15 (2001) 675–685.
- [13] J. Eberly, J. Van Mieghem, Multi-factor dynamic investment under uncertainty, *J. Econ. Theory* 75 (1997) 345–387.
- [14] S. Grenadier, Option exercise games: an application to the equilibrium investment strategies of firms, *Rev. Finan. Stud.* 15 (2002) 691–721.
- [15] X. Guo, An explicit solution to an optimal stopping problem with regime switching, *J. Appl. Prob.* 38 (2001) 464–481.
- [16] X. Guo, Inside information and stock fluctuations, Ph.D. Dissertation, Rutgers University, 1999.
- [17] J. Hamilton, A new approach to the economic analysis of nonstationary time series and the business cycle, *Econometrica* 57 (1989) 357–384.
- [18] T. Harchaoui, P. Lasserre, Testing the option value theory of irreversible investment, *Int. Econ. Rev.* 42 (2001) 141–166.
- [19] M. Harrison, T. Sellke, A. Taylor, Impulse control of Brownian motion, *Math. Operations Res.* 8 (1983) 454–466.
- [20] F. Hayashi, Tobin's marginal  $q$  and average  $q$ : a neoclassical interpretation, *Econometrica* 50 (1982) 213–224.
- [21] D. Jorgenson, Capital theory and investment behavior, *Amer. Econ. Rev. Papers Proc.* 53 (1963) 247–259.
- [22] I. Karatzas, S. Shreve, *Brownian Motion and Stochastic Calculus*, 2nd Edition, Springer, New York, 1991.
- [23] F. Kydland, E. Prescott, Time to build and aggregate fluctuations, *Econometrica* 50 (1982) 1345–1370.
- [24] R. McDonald, D. Siegel, The value of waiting to invest, *Quart. J. Econ.* 101 (1986) 707–728.
- [25] E. Morellec, Asset liquidity, capital structure and secured debt, *J. Finan. Econ.* 61 (2001) 173–206.
- [26] R. Pindyck, Irreversible investment, capacity choice, and the value of the firm, *Amer. Econ. Rev.* 78 (1988) 969–985.