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Optimal Selling Rules in a Regime Switching Model

X. Guo and Q. Zhang

***Abstract*—In this note, we derive optimal selling rules under a regime switching model. The optimal stopping rule is of a threshold type for each state, derived via the "modified smooth fit." The proof is via the martingale theory. Numerical examples are reported to demonstrate the dependence of threshold levels with various parameters and to compare our result with some suboptimal selling rules.**

***Index Terms*—Markov process, optimal stopping, regime switching, smooth fit.**

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I. Introduction

Consider an investor who holds a share of stock whose price at time t is $X(t)$. The investor's goal is to find the "best" selling time τ to maximize the discounted expected payoff, i.e., the value function is $\max_{\tau\le\infty} E_x[e^{-r\tau}(X_\tau - K)]$, where K is the amount to be paid back when the investor sells the stock, $X(0)=x$, and r is the discount factor (e.g., due to inflation). Assume also that the investor is not clairvoyant. This is an optimal stopping time problem with the stopping time $\tau \le t$ measurable to the "information" available up to time t (characterized by the σ-algebra generated by $X(s), 0 \le s \le t$). Clearly, the choice of an optimal stopping rule will be dictated by the underlying model for $X(\cdot)$.

This problem was first studied by McKean [6] in the 1960s. Assuming that $X(t)$ follows a geometric Brownian motion (now well known as the Black–Scholes model) such that $dX(t) = \mu X(t)dt + \sigma X(t)dW(t)$, and with $(X_\tau - K)$ modified equivalently by $(X_\tau - K)^+$, he solved the problem explicitly. He showed that if $\mu > r$, then one can patiently "wait and see" and the value function is infinite; if $\mu < r$, then there exists an $x^* = x^*(\mu, r, \sigma, K)$ so that the optimal stopping time $\tau^* = \inf\{t > 0, X(t) \ge x^*\}$; $\mu = r$ is a degenerate case for which the value function is x, the initial value of $X(t)$, with the corresponding $\tau^* = \infty$, a.s.

If we assume more realistically that the market fluctuates between "bull" and "bear" states and that μ and σ take different values in different market states, then the problem of this investor can be cast in a regime switching model, or the Markov-modulated Brownian motion model. More precisely, we consider the following switching diffusion process:

$$dX(t) = X(t)\mu_{\epsilon(t)}dt + X(t)\sigma_{\epsilon(t)}dW(t), \qquad X(0) = x \tag{1}$$

where $\epsilon(t) \in \{1, 2, \ldots, S\}$ is a finite-state continuous-time Markov chain, and $W(t)$ is a standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, P)$. Here, we assume that $W(\cdot)$ and $\epsilon(\cdot)$ are independent, and μ_i and σ_i are known parameters for any given $\epsilon(t) = i$.

Note that McKean's solution corresponds to the special case when the parameters μ and σ, respectively, are identical in different states.

This regime switching model has been studied by many researchers in various contexts; see, for example, [1], [3], and [9]. The work in [3] dealt with perpetual lookback option pricing and solved a related optimal stopping time problem by extending the well-known technique of smooth fit. The work in [9] used a two-point-boundary-value approach and provided a suboptimal selling rule involving target and cut-loss levels, which is optimal in that particular context.

In this note, we exploit the techniques developed in [3] to derive an optimal selling rule for this investor, assuming (1). For explicitness, we assume that $S = 2$ and the generator of $\epsilon(t)$ is of the form $\begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}$, with $\lambda_1, \lambda_2 > 0$. We derive explicitly a general optimal selling rule and the corresponding value function in a closed-form. We show that when $r < B_1$ $(=\{\mu_1 - \lambda_1 + \mu_2 - \lambda_2 + \sqrt{[(\mu_1 - \lambda_1) - (\mu_2 - \lambda_2)]^2 + 4\lambda_1\lambda_2}\}/2)$, it is optimal to wait and get an infinite return; when $r > \max(\mu_1, \mu_2) \ge B_1$, the optimal stopping rule is of a threshold type for each state; $r = B_1$ is the degenerate case for which the value is $x = X(0)$ and the waiting time is infinite. (The case of $r \in (B_1, \max(\mu_1, \mu_2))$ remains unsolved). The proof of optimality is via the Dynkin's formula and local martingales. Finally, we numerically illustrate the dependence of our optimal threshold levels on various parameters and the difference between our solution and that in [9].

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II. MAIN RESULTS

Consider the following optimal stopping problem with a state–space (x, ϵ):

$$V_i^*(x) = \sup_{0 \le \tau \le \infty} E[e^{-r\tau}(X(\tau) - K) \,|\, X(0) = x, \epsilon(0) = i] \quad (2)$$

where τ is an $\mathcal{F}_t = \sigma\{(W(s), \epsilon(s)) \,|\, s \le t\}$-stopping time.

Here, we assume $x > 0$ and exclude the case of $x = 0$ for which $V^* = 0$ and $\tau = \infty$.

Intuitively, if r is very small so that $e^{rt} < E[X_t]$, then it does not hurt to wait and the payoff is infinite. This intuition can be formalized.

Lemma 1: When $\lambda_i > 0, (i = 1, 2)$

$$\begin{aligned} M_i(t) &= E\left[\exp \int_0^t \mu_\epsilon(s)\, ds \,|\, \epsilon(0) = i\right] \\ &= \frac{\mu_i - B_2}{B_1 - B_2} e^{B_1 t} + \frac{B_1 - \mu_i}{B_1 - B_2} e^{B_2 t}. \end{aligned} \quad (3)$$

If $\lambda_i = 0$, then $M_i(t) = \exp(\mu_i t)$, and for $i \ne j, j = 1, 2$

$$M_j(t) = \frac{(\mu_i - \mu_j)\exp(\mu_j - \lambda_j)t}{\mu_i - \mu_j + \lambda_j} + \frac{\lambda_j \exp \mu_i t}{\mu_i - \mu_j + \lambda_j}.$$

Here, $B_1 > B_2$ are the roots

$$x^2 - (\mu_1 - \lambda_1 + \mu_2 - \lambda_2)x + (\mu_1 - \lambda_1)(\mu_2 - \lambda_2) - \lambda_1\lambda_2 = 0$$

such that

$$B_{1,2} = \frac{\mu_1 - \lambda_1 + \mu_2 - \lambda_2 \pm \sqrt{[(\mu_1 - \lambda_1) - (\mu_2 - \lambda_2)]^2 + 4\lambda_1\lambda_2}}{2}. \quad (4)$$

Proof: Starting from time 0 and state i, between time 0 and Δt, there is either at least one regime shift (RS) of $\epsilon(t)$ from state i to $j \ne i$, with probability $1 - e^{-\lambda_i \Delta t}$; or no RS and ϵ remains in state i with probability $e^{-\lambda_i \Delta t}$ and the process starts afresh again. It is clear that

$$\begin{aligned} M_i(t) &= E\left[\exp \int_0^t \mu_\epsilon(s)\, ds \left\{1_{(\text{at least one RS})} + 1_{(\text{no RS})}\right\}\right] \\ &= P\{\text{no RS}\} \exp(\mu_i \Delta t) M_i(t - \Delta t) \\ &\quad + P\{\text{at least one RS}\} M_j(t + \eta\Delta t) + O((\Delta t)^2) \\ &= \exp(-\lambda_i \Delta t)\exp(\mu_i \Delta t) M_i(t - \Delta t) \\ &\quad + (1 - e^{-\lambda_i \Delta t}) M_j(t + \eta \Delta t) + O((\Delta t)^2) \end{aligned}$$

with $i \ne j$, $|\eta| \le 1$.

By a first-order Taylor's expansion on e^x and some algebra, we get the corresponding ordinary differential equation (ODEs) with initial conditions for $M_i(t)$ such that

$$\lambda_i M_j(t) = -(\mu_i - \lambda_i) M_i(t) + M_i'(t)$$

with $M_i(0) = 1$, $M_i'(0) = \mu_i$, $i \ne j, i, j = 1, 2$. This implies that

$$\begin{aligned} M_i''(t) &- (\mu_i - \lambda_i + \mu_j - \lambda_j) M_i'(t) \\ &+ ((\mu_i - \lambda_i)(\mu_j - \lambda_j) - \lambda_i\lambda_j) M_i = 0 \end{aligned}$$

with the initial conditions $M_i(0) = 1, M_i'(0) = \mu_i$.

Solving this second-order ODE with the form of $M_i(t) = \exp(Bt)$ gives us

$$M_i(t) = A_{i1}\exp(B_1 t) + A_{i2}\exp(B_2 t).$$

Here, the coefficients are uniquely determined by the boundary conditions.

The degenerate case when $\lambda_1\lambda_2 = 0$ can be solved in a similar fashion. This completes the proof. □

A. Case 1: $r \le B_1$

First, note that $X(t)$ in (1) can be written as

$$X(t) = X(0)\exp\left(\int_0^t \left(\mu_{\epsilon(s)} - \frac{\sigma_{\epsilon(s)}^2}{2}\right) ds + \int_0^t \sigma_{\epsilon(s)}\, dW(t)\right).$$

In view of this, $\exp(\int_0^t \sigma_{\epsilon(s)}\, dW(s) - (1/2)\int_0^t \sigma_{\epsilon(s)}^2\, ds)$ defines a martingale. Moreover, it is clear from Lemma 1 that $\{e^{-rt}X_t\}_{t \ge 0}$ is submartingale if $B_1 > r$, and supermartingale if $B_1 < r$. Furthermore

$$\lim_{t\to\infty} E[e^{-rt}X_t] = \lim_{t\to\infty} e^{(B_1 - r)t}X_0 = \begin{cases} \infty, & \text{if } B_1 > r \\ x, & \text{if } B_1 = r \\ 0, & \text{if } B_1 < r \end{cases} \quad (5)$$

and for $T < \infty$ and any stopping time τ

$$E\left[e^{-r(\tau\wedge T)}X_{\tau\wedge T}\right] - e^{-r(\tau\wedge T)}K \le E\left[e^{-r(\tau\wedge T)}X_{\tau\wedge T}\right]. \quad (6)$$

Therefore, it is clear that if $B_1 > r$, $\tau^* = \infty$ and the value function is infinite. When $B_1 = r$, $\tau^* = \infty$ and the value function is x, following from the optional sampling theorem.

Theorem 1: For $\lambda_1 > 0, \lambda_2 > 0$, $V_i^*(x) < \infty$ if and only if $r \ge B_1$. In particular, when $r = B_1$, $V_i^*(x) = x$.

Remark 1: Note that the necessary and sufficient conditions for Theorem 1 hold only when $\lambda_1\lambda_2 \ne 0$. When $\lambda_1\lambda_2 = 0$, it is possible that $\min(V_1(x), V_2(x)) < \infty = \max(V_1(x), V_2(x))$. One can easily construct an example with $\lambda_1 = 0$ and $r \in (\mu_1, \mu_2 - \lambda_2)$.

B. Case 2: $r > \max(\mu_1, \mu_2) \ge B_1$

In this case, the value function is finite and is determined by the optimal stopping time τ. If we focus our attention on the threshold type stopping rules (whose optimality is to be verified), then it is reasonable to anticipate that the thresholds should be different according to the state of ϵ.

Denoting each threshold x_i for the state i, we first provide some intuitive derivation of such x_i, and then prove its optimality via the martingale theory.

1) Case $x_1 < x_2$: We first consider the case $x_1 < x_2$. Suppose at time $0, \epsilon(0) = i$. The very definition of x_i implies that if $x > x_i$, then one should stop if it is in state i and, therefore, $V_i(x) = x - K$; otherwise, one should hold the stock. Moreover, between $(0, \delta t)$, ϵ may change to state j with probability $\lambda_i \delta t$ for which the value function is $V_j(X(\delta t))$; and with probability $1 - \lambda_i\delta t$, ϵ remains at i for which the value function is $V_i(X(\delta t))$. That is

$$V_i(x) \le e^{-r\delta t}\{\lambda_i \delta t V_j(X(\delta t)) + (1 - \lambda_i\delta t)V_i(X(\delta t))\}.$$

This inequality becomes an equality if the strategy be optimal.

Now, assuming the value functions are sufficiently smooth, Ito's rule yields, for $x \in [x_1, x_2]$

$$\begin{cases} (r+\lambda_2)V_2(x) = x\mu_2 V_2'(x) + \frac{1}{2}x^2\sigma_2^2 V_2''(x) + \lambda_2(x-K), \\ V_1(x) = x - K \end{cases} \tag{7}$$

for $x \in [0, x_1)$

$$\begin{cases} (r+\lambda_1)V_1(x) = x\mu_1 V_1'(x) + \frac{1}{2}x^2\sigma_1^2 V_1''(x) + \lambda_1 V_2(x) \\ (r+\lambda_2)V_2(x) = x\mu_2 V_2'(x) + \frac{1}{2}x^2\sigma_2^2 V_2''(x) + \lambda_2 V_1(x) \end{cases} \tag{8}$$

and for $x \in [x_2, \infty]$, $V_1(x) = V_2(x) = x - K$.

Solving V_2 as a function of V_1 from the first equation in (8) and substituting it into the second equation lead to a fourth-order differential equation with constant coefficients after a logarithm transformation. In view of this, the characteristic equation associated with (8) is

$$g_1(\beta)g_2(\beta) = \lambda_1\lambda_2 \tag{9}$$

where $g_i(\beta) = \lambda_i + r - (\mu_i - (1/2)\sigma_i^2)\beta - (1/2)\sigma_i^2\beta^2, (i=1,2)$. It is not difficult to see that this equation has four distinct roots $\beta_1 > \beta_2 > 0 > \beta_3 > \beta_4$.

Therefore, the general form of the solution to (8) is

$$\begin{aligned} V_1(x) &= \sum_{i=1}^{4} A_i x^{\beta_i} \\ V_2(x) &= \sum_{i=1}^{4} l_i A_i x^{\beta_i} \end{aligned} \tag{10}$$

with $l_i = l(\beta_i) = g_1(\beta_i)/\lambda_1 = \lambda_2/g_2(\beta_i)$, for $i = 1, 2, 3, 4$. Note that $V_1(0) = V_2(0) = 0$, implying that the negative power of x should be eliminated such that

$$\begin{aligned} V_1(x) &= A_1 x^{\beta_1} + A_2 x^{\beta_2}, \\ V_2(x) &= l_1 A_1 x^{\beta_1} + l_2 A_2 x^{\beta_2}. \end{aligned} \tag{11}$$

Next, converting

$$(r+\lambda_2)V(x) = x\mu_2 V'(x) + \frac{1}{2}x^2\sigma_2^2 V''(x) + \lambda_2(x-K) \tag{12}$$

to a linear ODE via a change of variables $x = e^y$ leads to

$$V_2(x) = C_1 x^{\gamma_1} + C_2 x^{\gamma_2} + \phi(x) \tag{13}$$

where $\phi(x)$ is a special solution to (12), and γ_i $(i=1,2)$ are the real roots of

$$\mu_2\gamma + \frac{1}{2}\sigma_2^2\gamma(\gamma-1) = r + \lambda_2. \tag{14}$$

In particular, if $r + \lambda_2 - \mu_2 \neq 0$, one can take

$$\phi(x) = -\frac{\lambda_2 K}{r+\lambda_2} + \frac{\lambda_2 x}{r+\lambda_2-\mu_2}. \tag{15}$$

In order to uniquely determine $V_1(x)$ and $V_2(x)$, we must solve for A_1, A_2, C_1, C_2, x_1, and x_2. To this end, we need apply the smooth fit principle.

Smooth fit along the boundaries (i.e., at $x = x_1$ and $x = x_2$) yields $V_1(x+) = V_1(x-)$ and $V_1'(x+) = V_1'(x-)$ so that

$$\begin{cases} A_1 x_1^{\beta_1} + A_2 x_1^{\beta_2} = x_1 - K, \\ \beta_1 A_1 x_1^{\beta_1} + \beta_2 A_2 x_1^{\beta_2} = x_1. \end{cases} \tag{16}$$

Smoothness of $V_2(x)$ at x_1 and x_2 (i.e., the "modified smooth fit" as in [3]) suggests

$$\begin{aligned} l_1 A_1 x_1^{\beta_1} + l_2 A_2 x_1^{\beta_2} &= C_1 x_1^{\gamma_1} + C_2 x_1^{\gamma_2} + \phi(x_1) \\ l_1\beta_1 A_1 x_1^{\beta_1} + l_2\beta_2 A_2 x_1^{\beta_2} &= \gamma_1 C_1 x_1^{\gamma_1} + \gamma_2 C_2 x_1^{\gamma_2} + x_1\phi'(x_1) \end{aligned} \tag{17}$$

hence

$$\begin{aligned} C_1 x_2^{\gamma_1} + C_2 x_2^{\gamma_2} + \phi(x_2) &= x_2 - K \\ \gamma_1 C_1 x_2^{\gamma_1} + \gamma_2 C_2 x_2^{\gamma_2} - x_2\phi'(x_2) &= x_2. \end{aligned} \tag{18}$$

Combining these three equations with some algebraic manipulations, we obtain

$$\begin{pmatrix} x_1^{-\gamma_1} & 0 \\ 0 & x_1^{-\gamma_2} \end{pmatrix} F_1(x_1, \phi(x_1)) = \begin{pmatrix} x_2^{-\gamma_1} & 0 \\ 0 & x_2^{-\gamma_2} \end{pmatrix} F_2(x_2, \phi(x_2)) \tag{19}$$

where

$$\begin{aligned} F_1(x, g(x)) = &\begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \left[\begin{pmatrix} l_1 & l_2 \\ l_1\beta_1 & l_2\beta_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{pmatrix}^{-1} \right. \\ &\left. \times \begin{pmatrix} x-K \\ x \end{pmatrix} - \begin{pmatrix} g(x) \\ xg'(x) \end{pmatrix} \right] \end{aligned} \tag{20}$$

and

$$F_2(x, g(x)) = \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \begin{pmatrix} x-K-g(x) \\ x - xg'(x) \end{pmatrix}. \tag{21}$$

2) Case $x_1 > x_2$: The analysis is identical to the case of $x_1 < x_2$ with corresponding notational changes. We denote $\tilde{\phi}(x)$ as a special solution to

$$(r+\lambda_1)V(x) = x\mu_1 V'(x) + \frac{1}{2}x^2\sigma_1^2 V''(x) + \lambda_1(x-K) \tag{22}$$

$\tilde{\gamma}_i$ $(i=1,2)$ as the real roots of

$$\mu_1\gamma + \frac{1}{2}\sigma_1^2\gamma(\gamma-1) = r + \lambda_1. \tag{23}$$

Accordingly F_1 and F_2 are replaced, respectively, by

$$\begin{aligned} \tilde{F}_1(x, g(x)) = &\begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \left[\begin{pmatrix} \tilde{l}_1 & \tilde{l}_2 \\ \tilde{l}_1\beta_1 & \tilde{l}_2\beta_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{pmatrix}^{-1} \right. \\ &\left. \times \begin{pmatrix} x-K \\ x \end{pmatrix} - \begin{pmatrix} g(x) \\ xg'(x) \end{pmatrix} \right] \end{aligned} \tag{24}$$

with $\tilde{l}_i = 1/l_i$ and

$$\tilde{F}_2(x, g(x)) = \begin{pmatrix} 1 & 1 \\ \tilde{\gamma}_1 & \tilde{\gamma}_2 \end{pmatrix}^{-1} \begin{pmatrix} x-K-g(x) \\ x - xg'(x) \end{pmatrix}. \tag{25}$$

3) Case $x_1 = x_2 = x^*$: It is not difficult to show that the value function reduces to the McKean's problem in this case. Recall that

$$\begin{aligned} V_1(x) &= A_1 x^{\beta_1} + A_2 x^{\beta_2} \\ V_2(x) &= l_1 A_1 x^{\beta_1} + l_2 A_2 x^{\beta_2} \end{aligned}$$

for $x \in [0, x^*]$ and $V_1(x) = V_2(x) = x - K$ for $x \geq x^*$. Smooth fit conditions lead to

$$\begin{cases} A_1(x^*)^{\beta_1} + A_2(x^*)^{\beta_2} = x^* - K \\ \beta_1 A_1(x^*)^{\beta_1} + \beta_2 A_2(x^*)^{\beta_2} = x^* \end{cases} \tag{26}$$

and

$$\begin{cases} l_1 A_1(x^*)^{\beta_1} + l_2 A_2(x^*)^{\beta_2} = x^* - K \\ l_1\beta_1 A_1(x^*)^{\beta_1} + l_2\beta_2 A_2(x^*)^{\beta_2} = x^*. \end{cases} \tag{27}$$

Necessarily, we have $A_1 = l_1 A_1$ and $A_2 = l_2 A_2$, meaning $V_1 = V_2$ and $\mu_1 = \mu_2, \sigma_1 = \sigma_2$ from (8). The value function can then be easily rederived via the smooth fit, which is exactly the solution of McKean.

We summarize our results as follows. For ease of exposition, we use Y_1, Y_2 to denote column vectors and define

$$H(x_1, x_2, Y_1, Y_2) = \begin{pmatrix} x_1^{-\gamma_1} & 0 \\ 0 & x_1^{-\gamma_2} \end{pmatrix} Y_1 - \begin{pmatrix} x_2^{-\gamma_1} & 0 \\ 0 & x_2^{-\gamma_2} \end{pmatrix} Y_2 \tag{28}$$

and

$$\tilde{H}(x_1, x_2, Y_1, Y_2) = \begin{pmatrix} x_1^{-\tilde{\gamma}_1} & 0 \\ 0 & x_1^{-\tilde{\gamma}_2} \end{pmatrix} Y_1 - \begin{pmatrix} x_2^{-\tilde{\gamma}_1} & 0 \\ 0 & x_2^{-\tilde{\gamma}_2} \end{pmatrix} Y_2. \tag{29}$$

Theorem 2: Assume $r > \max(\mu_1, \mu_2)$, and $\mu_1 \neq \mu_2$ or $\sigma_1 \neq \sigma_2$.

Case 1: If there exists an $x_1 < x_2$ such that $H(x_1, x_2, F_1(x_1, \phi(x_1)), F_2(x_2, \phi(x_2))) = 0$ and $x_2 \geq rK/(r - \max(\mu_1, \mu_2))$, then $V_i^*(x) = V_i(x)$, if $V_i(x) > x - K$ with

$$V_1(x) = \begin{cases} A_1 x^{\beta_1} + A_2 x^{\beta_2}, & \text{if } x < x_1 \\ x - K, & \text{if } x \geq x_1 \end{cases}$$
$$V_2(x) = \begin{cases} l_1 A_1 x^{\beta_1} + l_2 A_2 x^{\beta_2}, & \text{if } x < x_1 \\ C_1 x^{\gamma_1} + C_2 x^{\gamma_2} + \phi(x), & \text{if } x_2 > x \geq x_1 \\ x - K, & \text{if } x \geq x_2 \end{cases} \tag{30}$$

where $\beta_1 > \beta_2 > 0$ are roots of (9)

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} x_1^{\beta_1} & x_1^{\beta_2} \\ \beta_1 x_1^{\beta_1} & \beta_2 x_1^{\beta_2} \end{pmatrix}^{-1} \begin{pmatrix} x_1 - K \\ x_1 \end{pmatrix}$$
$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} x_2^{\gamma_1} & x_2^{\gamma_2} \\ \gamma_1 x_2^{\gamma_1} & \gamma_2 x_2^{\gamma_2} \end{pmatrix}^{-1} \begin{pmatrix} x_2 - K - \phi(x_2) \\ x_2 - x_2\phi'(x_2) \end{pmatrix}.$$

Case 2: If there exists an $x_1 > x_2$ such that $\tilde{H}(x_1, x_2, \tilde{F}_2(x_1, \tilde{\phi}(x_1)), \tilde{F}_1(x_2, \tilde{\phi}(x_2))) = 0$ and $x_1 \geq rK/(r - \max(\mu_1, \mu_2))$, then $V_i^*(x) = V_i(x)$ if $V_i(x) \geq x - k$ with

$$V_1(x) = \begin{cases} \tilde{A}_1 x^{\beta_1} + \tilde{A}_2 x^{\beta_2}, & \text{if } x < x_2 \\ \tilde{C}_1 x^{\tilde{\gamma}_1} + \tilde{C}_2 x^{\tilde{\gamma}_2} + \tilde{\phi}(x), & \text{if } x_1 > x \geq x_2 \\ x - K, & \text{if } x \geq x_1 \end{cases}$$
$$V_2(x) = \begin{cases} l_1 \tilde{A}_1 x^{\beta_1} + l_2 \tilde{A}_2 x^{\beta_2}, & \text{if } x < x_2 \\ x - K, & \text{if } x \geq x_2 \end{cases} \tag{31}$$

where

$$\begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} = \begin{pmatrix} l_1 x_2^{\beta_1} & l_2 x_2^{\beta_2} \\ l_1\beta_1 x_2^{\beta_1} & l_2\beta_2 x_2^{\beta_2} \end{pmatrix}^{-1} \begin{pmatrix} x_2 - K \\ x_2 \end{pmatrix}$$

and

$$\begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix} = \begin{pmatrix} x_1^{\tilde{\gamma}_1} & x_1^{\tilde{\gamma}_2} \\ \tilde{\gamma}_1 x_1^{\tilde{\gamma}_1} & \tilde{\gamma}_2 x_1^{\tilde{\gamma}_2} \end{pmatrix}^{-1} \begin{pmatrix} x_1 - K - \tilde{\phi}(x_1) \\ x_1 - x_1\tilde{\phi}'(x_1) \end{pmatrix}.$$

Furthermore, let $D = \{(x, i) \,|\, V_i(x) > x - K\}$. The optimal stopping rule for both cases is given by $\tau^* = \inf\{t > 0 \,|\, (X(t), \epsilon(t)) \notin D\}$.

In particular, if $\mu_1 = \mu_2, \sigma_1 = \sigma_2$, then $x_1 = x_2 = x^*$, and

$$V_i^*(x) = V_1(x) = V_2(x) = \begin{cases} \dfrac{(x^*)^{1-\beta}}{\beta} x^{\beta}, & \text{if } x < x^* \\ x - K, & \text{if } x \geq x^* \end{cases}$$

where $\beta > 0$ satisfies $r - (\mu_i - (1/2)\sigma_i^2)\beta - (1/2)\sigma_i^2\beta^2 = 0$ and $x^* = \beta K/(\beta - 1))(> rK/(r - \mu_i))$, and $\gamma_i, \tilde{\gamma}, \tilde{\phi}, \phi$, are given by (14), (23), (15), and (22), respectively.

One can prove the optimality of the value function by modifying the verification theorem argument in [3]. Here, we adopt a slightly different approach.

Proof of Theorem 2: It is easy to see that $V_i(\infty) = \infty$, $i = 1, 2$, and

$$D = \{(x, 1) \,|\, x \in (0, x_1)\} \cup \{(x, 2) \,|\, x \in (0, x_2)\}.$$

For any $v(x, i) \in C^2$, define

$$\mathcal{L}v(x, i) = x\mu_i \frac{\partial v(x, i)}{\partial x} + \frac{1}{2} x^2 \sigma_i^2 \frac{\partial^2 v(x, i)}{\partial x^2} + \lambda_i (v(x, 3 - i) - v(x, i)) - rv(x, i).$$

Take $v(x, i) = V_i(x)$. Then it is easy to check that the following dynamic programming principle holds:

$$\max(\mathcal{L}v, x - K - v) = 0.$$

Note that $V_i(x)$ is C^1 and not necessarily C^2 at x_1—this is remedied by the smooth approximation. Let

$$\phi_0(x) = \begin{cases} e^{-1/(1-x^2)}, & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1. \end{cases}$$

Define $\phi(x) = \phi_0(x) / \int_{-\infty}^{\infty} \phi_0(x)\,dx$. Then, it is easy to check that $\phi(x) \in C_0^{\infty}$ and $\int_{-\infty}^{\infty} \phi(x)\,dx = 1$. Denote $k_\eta(x)$ to be the kernel function $k_\eta(x) = (1/\eta)\phi(x/\eta)$ and let $v^\eta(x, i) = (k_\eta * v)(x, i)$ with $v(x, i) = V_i(x)$ be the corresponding convolution. Then, $v^\eta(x, i) \in C^\infty$, $\eta > 0$. Moreover, as $\eta \to 0$, we can show as in Øksendal [7, Th. D.1] that: 1) $v^\eta \to v$ uniformly on compact sets in R; 2) $\mathcal{L}v^\eta \to \mathcal{L}v$ uniformly on compact sets in $(\partial D)^c$; and 3) $\{\mathcal{L}v^\eta\}$ is locally bounded on R. With these properties, one can show as in [7] that, for any stopping time τ,

$$\begin{aligned} v(x, i) &\geq E\left[e^{-r(t\wedge\tau)} v(X(t\wedge\tau), \epsilon(t\wedge\tau))\right] \\ &\geq E\left[e^{-r(t\wedge\tau)} (X(t\wedge\tau) - K)\right]. \end{aligned} \tag{32}$$

Therefore

$$v(x, i) \geq E[e^{-r\tau} v(X(\tau), \epsilon(\tau))] \geq E[e^{-r\tau}(X(\tau) - K)] \tag{33}$$

from the uniform integrability of $e^{-rt} v(X(t), \epsilon(t))$. To show the optimality of τ^*, note that if $\tau^* < \infty$, a.s., then $v(X(\tau^*), \epsilon(\tau^*)) = X(\tau^*) - K$. In this case, Dynkin's formula yields $v(x, i) = E[e^{-r\tau^*}(X(\tau^*) - K)]$. Otherwise, let $D_k = D \cap \{(x, i) : 1/k < x < k, i = 1, 2\}$, for $k = 1, 2, \ldots$, and let $\tau_k = \inf\{t \geq 0 \,|\, (X(t), \epsilon(t)) \notin D_k\}$. Then clearly $\tau_k \to \tau^*$ a.s. Moreover, as in [9, Ths. 4.5 and 4.6], we see that, for each k, $\tau_k < \infty$ a.s. Next, by the definition of τ_k, we have $v(X(\tau_k), \epsilon(\tau_k)) = v(X(\tau_k), \epsilon(\tau_k)) I_{\{X(\tau_k) = k\}} + v(X(\tau_k), \epsilon(\tau_k)) I_{\{X(\tau_k) < k\}}$. Note that $v(X(\tau_k), \epsilon(\tau_k)) I_{\{X(\tau_k) < k\}} = (X(\tau_k) - K) I_{\{X(\tau_k) < k\}} \leq X(\tau_k) - K$. Note also that $e^{-r\tau_k} I_{\{X(\tau_k) = k\}} \to 0$, as $k \to \infty$, a.s. It follows that $E[e^{-r\tau_k} v(X(\tau_k), \epsilon(\tau_k)) I_{\{X(\tau_k) = k\}}] \to 0$. Therefore, we have $v(x, i) \leq E[e^{-r\tau_k} v(X(\tau_k), \epsilon(\tau_k))] \to E[e^{-r\tau^*}(X(\tau^*) - K)]$, as

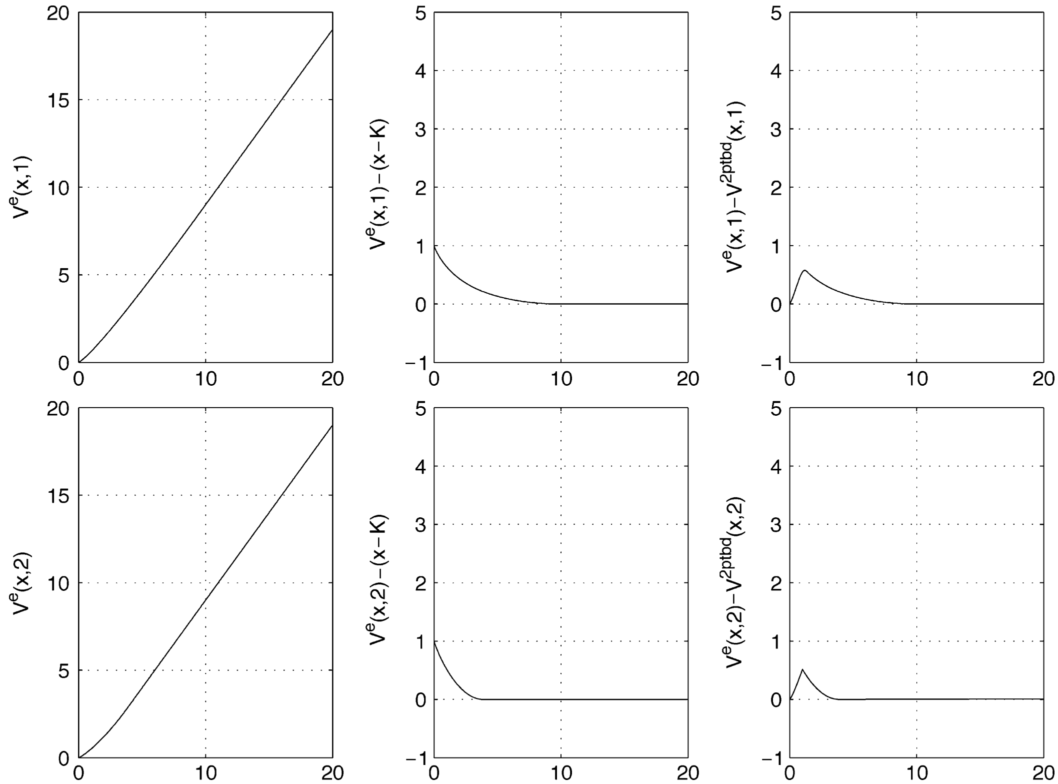

Fig. 1. Value functions and comparisons.

$k \to \infty$. This combined with (33) summaries the optimality of the value function, i.e., $v(x, i) = E[e^{-r\tau^*}(X(\tau^*) - K)]$. □

Remark 2: In order to apply Theorem 2, one needs to evaluate if the corresponding algebraic equations $H = 0$ or $\tilde{H} = 0$ admit solutions. This is easy to verify numerically, as shown in Section III.

Remark 3: It is nontrivial to prove *a priori* the modified smooth fit principle (hence, the regularity of the value function) for the regime switching model. This question is beyond the scope of this note.

Remark 4: The case of $r \in (B_1, \max(\mu_1, \mu_2))$ remains open and it is not clear to us if the value function will be finite in this case.

III. Numerical Simulation

As mentioned earlier, in [9] a two-point boundary value differential equation (TPBVDE) approach was exploited to derive an "optimal" selling rule for the threshold type stopping rules. This, however, is a suboptimal rule for our problem since the feasible solution was chosen from a constrained and smaller set of stopping rules. In this section we report numerical experiments for comparing our analytical solutions with the TPBVDE solutions.

First, we take $r = 3, \mu_1 = 2, \mu_2 = 1, K = 1, \lambda_1 = \lambda_2 = 5, \sigma_1 = 4, \sigma_2 = 2$, and examine our closed-form solutions and those by the TPBVDE method. (These numbers are so chosen to demonstrate the threshold levels. For related examples with real market data, see [8].) In this case, the threshold levels are given by $(x_1, x_2) = (9.97, 3.88)$. In Fig. 1, $V^{\mathrm{e}}(x, i)$ and $V^{2\mathrm{ptbd}}(x, i)$ denote value functions from our optimal stopping rule and from the TPBVDE approach, respectively. The difference between $V^e(x, i)$ and $(x - K)$ in Fig. 1 validates the basic structure of the optimal stopping policy in terms of threshold levels (x_1, x_2).

Next, we examine the monotonicity of these threshold levels with respect to σ_1, λ_1, and K.

TABLE I
Dependency on σ_1

σ_1	2	3	4	5
(x_1, x_2)	(5.02,3.31)	(7.10,3.60)	(9.97,3.88)	(13.74,4.15)

TABLE II
Dependency on λ_1

λ_1	3	4	5	6
(x_1, x_2)	(10.56,3.98)	(10.25,3.92)	(9.97,3.88)	(9.72,3.83)

TABLE III
Dependency on K

K	1	2	3	4
(x_1, x_2)	(9.98,3.88)	(19.95,7.76)	(29.97,11.62)	(39.95,15.50)

We first vary σ_1 and keep all other parameters fixed. The resulting (x_1, x_2) are listed in Table I. Both threshold levels x_1 and x_2 increase with the increase in σ_1. This shows that a larger σ_1 leads to a higher expected reward, and therefore a higher threshold levels.

We then vary λ_1. The result in Table II implies that both x_1 and x_2 decrease if λ_1 increases: this is because a larger λ_1 leads to a shorter period for $\epsilon(t)$ to stay at $\epsilon(t) = 1$ and a smaller weight on $\sigma_1 = 4$ $(>\sigma_2 = 2)$, which leads to a smaller average volatility.

We finally vary K. Table III suggests that both x_1 and x_2 increase in K due to the fact that a larger K implies a higher transaction cost which in turn needs to be compensated by a higher sample-wise return level.

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Iterative Learning Control for Systems With Input Deadzone

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***Abstract*—Most iterative learning control (ILC) schemes proposed hitherto were designed and analyzed without taking the input deadzone into account. Input deadzone is a kind of nonsmooth and nonaffine-in-input factor widely existing in actuators or mechatronics devices. It gives rise to extra difficulty due to the presence of singularity in the input channels. In this note, we disclose that ILC methodology remains effective for systems with input deadzone that could be nonlinear, unknown and state-dependent. Through rigorous proof, it is shown that despite the presence of the input deadzone, the simplest ILC scheme retains its ability of achieving the satisfactory performance.**

***Index Terms*—Convergence analysis, input deadzone, iterative learning control, nonlinear dynamics.**

NOMENCLATURE

$\mathcal{R}^n$ denotes the Euclidean space of dimension n; $\mathcal{R}$ denotes the set of real numbers; $\mathcal{Z}_+$ denotes the set of nonnegative integers; $|g|$ denotes the absolute value of g; $\|\mathbf{g}\|$ denotes the Euclidean norm of $\mathbf{g}$; $\mathcal{K} \triangleq \{0, 1, \ldots, N\}$ where N is a finite integer; $k \in \mathcal{K}$ denotes the time instance; $i \in Z_+$ denotes the iteration number; l_g denotes the Lipschitz constant of $\mathbf{g}(\mathbf{x}, k)$ or $g(\mathbf{x}, k)$; $\rho_c \triangleq \|\mathbf{c}\|$; $\rho_b \triangleq \max_{\mathbf{x},k} \|\mathbf{b}(\mathbf{x}, k)\|$; $\rho_u \triangleq \max_k |u_d(k)|$; $\rho_v \triangleq \max_k |v_d(k)|$; $\rho_{\eta_r} \triangleq \max_{\mathbf{x},k} |\eta_r(\mathbf{x}, k)|$; $\rho_m \triangleq \max_{\mathbf{x},k} \{m_l(\mathbf{x}, k), m_r(\mathbf{x}, k)\}$; In all notations, a quantity a associated with the subscripts i, D, r, i, and d belongs respectively to the left region of the deadzone, the deadzone, the right region of the deadzone, the real system at the i-th iteration, and the desired system; Let $* \in \{i, d\}$, $\eta_{l,*}(k) \triangleq \eta_l(\mathbf{x}_*, k)$, and $\eta_{r,*}(k) \triangleq \eta_r(\mathbf{x}_*, k)$ are the left and right boundary points of a deadzone; $m_{l,*}(k) \triangleq m_l(\mathbf{x}_*, k)$ and $m_{r,*}(k) \triangleq m_r(\mathbf{x}_*, k)$ are the left and right slopes of a deadzone; $I_{D,*} \triangleq [\eta_{l,*}(k), \eta_{r,*}(k)]$ is the deadzone; $I_{l,*} \triangleq (-\infty, \eta_{l,*}(k)]$ is the left region of the deadzone; $I_{r,*} \triangleq [\eta_{r,*}(k), +\infty)$ is the right region of the deadzone; $\gamma_{l,*} \triangleq 1 - \beta \mathbf{c}\mathbf{b}(\mathbf{x}_*, k) m_{l,*}(k)$, $\gamma_{r,*} \triangleq 1 - \beta \mathbf{c}\mathbf{b}(\mathbf{x}_*, k) m_{r,*}(k)$, $\mathbf{f}_i(k) \triangleq \mathbf{f}(\mathbf{x}_i, k)$, and $\mathbf{b}_i(k) \triangleq \mathbf{b}(\mathbf{x}_i, k)$.

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I. INTRODUCTION

In many industrial processes, often a task is repeated over a finite operation cycle and the perfect tracking is required from the very beginning. Iterative learning control (ILC) has been proposed to deal with this class of control problems. However, most ILC schemes proposed hitherto were designed and analyzed without taking the input deadzone into account. Input deadzone is a kind of nonsmooth and nonaffine-in-input factor widely existing in actuators or mechatronics devices. It gives rise to extra difficulty in control due to the presence of singularity in the input channels. The input deadzone problem has been studied by many researchers. Some useful techniques for overcoming deadzone are variable structure control and dithering. Motivated by pursuing better control performance, several adaptive inverse approaches were proposed [1]. Recently, soft computing such as fuzzy logic and neural network-based control algorithms has also been applied to handle problems relevant to deadzones [2], [3]. In these works, the input deadzone was assumed to be independent of the system operating conditions. Such an assumption does not hold when we deal with a control process with high precision requirement.

In this work, we disclose a finding that ILC schemes [4], [5], originally designed for systems without input deadzone, can effectively compensate the nonlinear deadzone through control repetitions. In a rigorous mathematical manner, we prove that, despite the presence of the input deadzone, the simplest ILC scheme retains its ability of achieving the satisfactory performance in tracking control. The nonlinear state-dependent deadzone presents a new challenging problem to control theory in general, and to ILC convergence analysis in particular. To address this issue, the learning convergence is derived in two phases. First, the learning convergence to the desired regions is derived. In this phase, we prove that the actual control input sequence will enter the correct region after a finite number of iterations, then stay forever. In the second phase, we will use the mathematical induction principle to prove the uniform convergence of the learning control input sequence.

This note is organized as follows. The nomenclature of this note are first listed. In Section II, the control problem is formulated. The regional learning convergence is proven in Section III and the asymptotic tracking convergence is analyzed in Section IV.

II. PROBLEM FORMULATION

Consider the following discrete-time dynamic system:

$$\begin{aligned} \mathbf{x}_i(k+1) &= \mathbf{f}(\mathbf{x}_i(k), k) + \mathbf{b}(\mathbf{x}_i(k), k) u_i(k) \\ u_i(k) &= DZ[v_i(k)] \\ y_i(k+1) &= \mathbf{c}\mathbf{x}_i(k+1) \end{aligned} \tag{1}$$

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