

SOME OPTIMAL STOPPING PROBLEMS WITH NONTRIVIAL BOUNDARIES FOR PRICING EXOTIC OPTIONS

XIN GUO,* *IBM*

LARRY SHEPP,** *Rutgers University*

Abstract

We solve the following three optimal stopping problems for different kinds of options, based on the Black–Scholes model of stock fluctuations. (i) The perpetual lookback American option for the running maximum of the stock price during the life of the option. This problem is more difficult than the closely related one for the Russian option, and we show that for a class of utility functions the free boundary is governed by a nonlinear ordinary differential equation. (ii) A new type of stock option, for a company, where the company provides a guaranteed minimum as an added incentive in case the market appreciation of the stock is low, thereby making the option more attractive to the employee. We show that the value of this option is given by solving a nonalgebraic equation. (iii) A new call option for the option buyer who is risk-averse and gets to choose, *a priori*, a fixed constant l as a ‘hedge’ on a possible downturn of the stock price, where the buyer gets the maximum of l and the price at any exercise time. We show that the optimal policy depends on the ratio of x/l , where x is the current stock price.

Keywords: Lookback options; Black–Scholes model; optimal stopping; Bellman equation; free boundary

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1. Problems and main results

Let X_t be a geometric Brownian motion with drift μ and σ , that is,

$$dX_t = X_t \mu dt + X_t \sigma dW_t, \quad (1)$$

where μ, σ are constants and W_t is the standard Wiener process. This is the most commonly used Black–Scholes model for stock fluctuations X_t (cf. [2] and the references therein for alternate models). Based on this model, we consider several free boundary problems related to option pricings.

In Sections 1.1–1.3, we present three different problems and their explicit solutions, along with heuristic derivations of the solutions. In Section 2, we provide martingale-based proofs of the theorems and a generalization of one of our results.

1.1. A perpetual lookback American option

An option is a security that gives its holder the right, but not the obligation, to sell or buy something of value (for example, a share of stock) on specific terms at a fixed instant T or an

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* Postal address: IBM T. J. Watson Research Center, PO Box 218, Yorktown Heights, NY 10598, USA.

Email address: xinguo@us.ibm.com

** Postal address: Department of Statistics, Rutgers University, Piscataway, NJ 08854, USA.

arbitrary time $t \leq T$ during a certain period of time $[0, T]$ in the future. Different types of options assign different payoff functions. For instance, the payoff of a certain lookback option may be dependent on the minimum or maximum stock price achieved during the life of the option. We consider one type of lookback option, whose payoff is the amount by which the maximum stock price achieved during the life ($T \leq \infty$) of the option exceeds a fixed (strike) price (say K). It has features of American options in that it gives the holder the choice of an arbitrary exercise time. However, unlike the standard American call option whose payoff depends on the difference between the spot price at the execution time and the strike price K , the payoff of this option involves the running maximum of the stock price up to its execution time. Therefore, we call it the *perpetual lookback American option*.

Now, given this lookback American option, what is its value? How can an option holder find an optimal strategy for exercising this option? These questions can be formulated as the following optimal stopping time problem:

$$V^*(x, s) = \sup_{0 \leq \tau \leq \infty} E[e^{-r\tau} c(S_\tau)],$$

where $S_t = \max_{0 \leq u \leq t} X_u$ is the running maximum of X_t , r is a constant, τ is a stopping time, meaning that no clairvoyance is allowed, and $c(x)$ is some utility function.

For ease of exposition, we will start our analysis by studying the special case when $c(S_\tau) = (S_\tau - K)^+$. We will give explicit answers to the optimal stopping problem for this particular case. At the end of the paper, we will extend our result to a more general form of utility function $c(x)$. In particular, we provide sufficient conditions on $c(x)$ under which similar optimal stopping rules can be derived (Section 2.3).

For the moment, we concentrate on

$$V^*(x, s) = \sup_{0 \leq \tau \leq \infty} E[e^{-r\tau} (S_\tau - K)^+].$$

This optimal stopping problem has a state space (x, s) with $x \leq s$. In general, the structure of the solution depends on the so-called ‘free boundary’ ($x = g(s)$) between the continuation region (where $x \geq g(s)$ and where it is optimal to ‘keep holding the option’) and the stopping region (where $x \leq g(s)$ and where the best strategy is to ‘exercise the option immediately’).

In the continuation region, assuming $V^*(x, s)$ is C^2 , we have the ‘Bellman equation’:

$$rV^*(x, s) = \mu x V_x^*(x, s) + \frac{1}{2} \sigma^2 x^2 V_{xx}^*(x, s).$$

This equation can also be derived via the infinitesimal generator \mathcal{A} of the geometric Brownian motion (X_t, t) ,

$$\mathcal{A}f(t, x) = \frac{\partial f}{\partial t} + \mu x \frac{\partial f}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}.$$

General solutions of this equation will be of the form

$$V^*(x, s) = A_0(s)x^{\gamma_0} + A_1(s)x^{\gamma_1}, \quad (2)$$

where γ_1, γ_0 are the solutions of $r = \mu\gamma + \sigma^2\gamma(\gamma - 1)/2$.

Now, how can we determine $A_0(s)$, $A_1(s)$, and $g(s)$? It is reasonable to imagine that there exists a threshold s^* such that when $s < s^*$ and $x \leq g(s)$, it pays to wait. Further thoughts reveal that if x is very close to 0 and $s > K$, which means the current price is very low and yet we have some positive payoff, then we should exercise the option immediately. If not, we

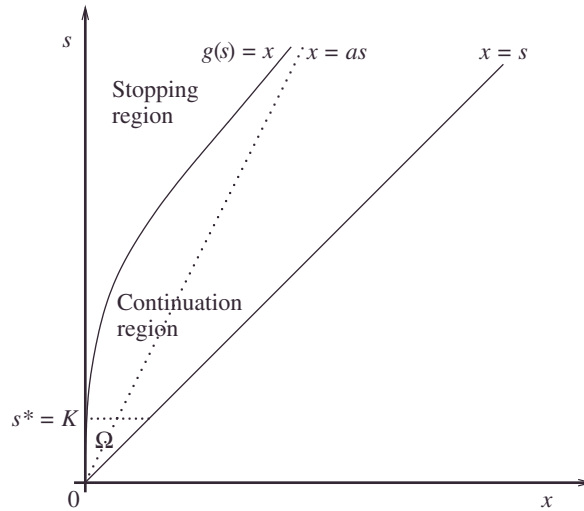


FIGURE 1.

may have to wait ‘too long’ for the price to bounce back and hence lose some appreciation for the cash value. Moreover, for any $s < K$, the current payoff is 0 and therefore it never hurts to wait. All these seem to suggest that $s^* = K$. As for the boundary $x = g(s)$, all we have in hand is the ‘principle of smooth fit’ (cf. [2] and the references therein) such that

$$V^*(g(s), s) = s - K \tag{3}$$

and

$$V_x^*(g(s), s) = 0. \tag{4}$$

Also, in order for the solution to be optimal, it is necessary to have

$$V_s^*(s, s) = 0. \tag{5}$$

This equation guarantees certain martingale properties for the value function. The equation $x = g(s)$ completely determines the value function $V^*(x, s)$ for all $s > K$.

Deriving from (2)–(5), we see that $x = g(s)$ satisfies

$$\gamma_0 \left(\frac{s}{g(s)} \right)^{\gamma_1} - \gamma_1 \left(\frac{s}{g(s)} \right)^{\gamma_0} = -\gamma_0 \gamma_1 (s - K) \frac{g'(s)}{g(s)} \left[\left(\frac{s}{g(s)} \right)^{\gamma_0} - \left(\frac{s}{g(s)} \right)^{\gamma_1} \right].$$

The value function in the bounded region Ω with boundary $s = K$, $x = s$, and $x = 0$ needs some extra care, however. When x approaches 0, this function is bounded, which suggests that $V^*(x, s) = A(s)x^{\gamma_0}$. Conditions such as $V_s^*(s, s) = 0$ and continuity along the line $s = K$ suffice to determine $V^*(x, s)$ in Ω (see Figure 1).

Trying to solve this optimal stopping problem based on our heuristic guess, we obtain the following answer and we prove in Section 2 that the solution is optimal.

Theorem 1. (i) *The value function $V^*(x, s)$ is finite if and only if $r > \max(0, \mu)$.*

(ii) *When $r > \max(0, \mu)$,*

$$V^*(x, s) = \begin{cases} s - K, & x \leq g(s), s > K, \\ \frac{s - K}{\gamma_0 - \gamma_1} \left(\gamma_0 \left(\frac{x}{g(s)} \right)^{\gamma_1} - \gamma_1 \left(\frac{x}{g(s)} \right)^{\gamma_0} \right), & s > x \geq g(s), s > K, \\ A \frac{\gamma_1}{\gamma_1 - \gamma_0} x^{\gamma_0}, & s < K, x < s, \end{cases}$$

where $0 \leq g(s) < as$ for all $s > K$ and $g(s)$ is the (unique) solution to the differential equation

$$\gamma_0 \left(\frac{s}{g(s)} \right)^{\gamma_1} - \gamma_1 \left(\frac{s}{g(s)} \right)^{\gamma_0} = -\gamma_0 \gamma_1 (s - K) \frac{g'(s)}{g(s)} \left[\left(\frac{s}{g(s)} \right)^{\gamma_0} - \left(\frac{s}{g(s)} \right)^{\gamma_1} \right], \quad (6)$$

such that when $s \rightarrow \infty$, $g(s)/s$ is asymptotic to a , that is, $\lim_{s \rightarrow \infty} g(s)/s = a$ and $0 \neq A = \lim_{s \rightarrow K^+} (s - K)/g(s)^{\gamma_0}$. Here, $\gamma_0 > 1 > 0 > \gamma_1$ are the solutions of

$$r = \mu\gamma + \frac{1}{2}\sigma^2\gamma(\gamma - 1)$$

and

$$a = \left(\frac{1 - 1/\gamma_1}{1 - 1/\gamma_0} \right)^{1/(\gamma_0 - \gamma_1)}. \quad (7)$$

The optimal stopping time τ^* is

$$\tau^* = \inf\{t \geq 0 \mid X_t = g(S_t), S_t > K\},$$

starting from $X_0 = x > 0, S_0 = s, x > g(s)$.

As the reader may realize, here the ‘free boundary’ $g(s)$ is delicate and the argument of its existence is quite subtle. In the next section, we will provide detailed discussion on the solution structure, as well as its proof with (elementary!) ODE techniques.

1.2. Does Microsoft need to issue new stock options?

Consider the following optimal stopping problem:

$$V^*(x, l) = \sup_{\tau} E[e^{-r\tau} (S_{\tau} - K)^+],$$

where $S_t = \max\{l, X_t\}$. This problem is equivalent to

$$V^*(x, s) = \sup_{\tau} E[e^{-r\tau} \max\{(X_{\tau} - K)^+, s\}],$$

where $s = l - K > 0$. Notice that this problem differs from the previous one in that there is no running maximum of X_t involved.

The motivation behind this problem is as follows: stock options (usually long-term American type) are given by many companies as a reward to its outstanding employees. A major drawback of this standard form of stock options is that its worth is ultimately determined by the stock price, and hence by the market value of the company. The latter is often tied to the market appreciation of certain sectors or industries. Thus, stock options may often fail to reflect the true value and

real performance of the company. This is exemplified by many well-established companies that cannot out-perform the market and consequently make their stock options essentially worthless and less appealing to employees. One alternative for the company, which is powerless to change the market, is to leverage the true value of its stock options in the following way: specify an l (slightly) bigger than the strike price K , such that the value of one share of option at the exercise date t is $(S_t - K)^+$, where $S_t = \max\{l, X_t\}$. From (4), we see it is equivalent to giving the outstanding employees a choice between some real cash and shares of stock options, without technically changing the standard form of stock options.

This new type of option has the following advantages: (i) employees with outstanding performance are guaranteed to get at least a certain amount of reward; (ii) the company does not need to reissue new shares of stock options when the market turns against their stocks. Therefore, this policy should greatly increase the morale of the company, as well as prevent further devaluations of their stocks by avoiding reissuing new shares of stock options. One possible drawback of this option is that it may cost the company dearly when the market has a downturn and which is precisely when the company may not be able to afford such options. So, the company should be very careful when issuing such an incentive. A proper choice of s is thus crucial for the business model of the company. For example, conservative companies may choose to assign a very low value of s . Consequently, finding a proper hedging strategy of this option is also an interesting problem.

Now, given this new type of stock option, what is the value of this option and what is the optimal strategy for exercising the option?

Again, the key is deciding the ‘free boundary’. The heuristics are rather intuitive: if the current value of the stock is very high (or too low), say, above a threshold $a^* > K$ (or below a threshold $b^* < K$), sell it immediately and get cash of $a^* - K$ (or $l - K$); otherwise, wait till the market is favorable.

Therefore, we have

$$\begin{aligned} rV^*(x, l) &= x\mu V_x^*(x, l) + \frac{1}{2}x^2\sigma^2 V_{xx}^*(x, l), & x \in (b^*, a^*), \\ V^*(x, l) &= x - K, & x \geq a^*, \\ V^*(x, l) &= l - K, & x \leq b^*, \end{aligned}$$

and the principle of smooth fit:

$$\begin{aligned} \left. \frac{V^*(x, l)}{\partial x} \right|_{x=a^*} &= 1, & \left. \frac{V^*(x, l)}{\partial x} \right|_{x=b^*} &= 0, \\ V^*(a^*, l) &= a^* - K, & V^*(b^*, l) &= l - K, \end{aligned}$$

which eventually lead to the next result.

Theorem 2. (i) When $r > \max(0, \mu)$, then $V^*(x, l) = V(x, l)$, where $V(x, l) \geq l - K$ is such that

$$V(x, l) = \begin{cases} l - K, & x \leq b^*, \\ (l - K) \left(\frac{\gamma_1}{\gamma_1 - \gamma_0} \left(\frac{x}{b^*}\right)^{\gamma_0} + \frac{\gamma_0}{\gamma_0 - \gamma_1} \left(\frac{x}{b^*}\right)^{\gamma_1} \right), & b^* < x < a^*, \\ x - K, & a^* \leq x, \end{cases}$$

with $a^* > l$ given by

$$\frac{a^* - K}{l - K} = \frac{\gamma_1}{\gamma_1 - \gamma_0} (c^*)^{\gamma_0} + \frac{\gamma_0}{\gamma_0 - \gamma_1} (c^*)^{\gamma_1}$$

and $b^* < l$ such that

$$b^* = \frac{(l - K)}{c^*(\gamma_0 - \gamma_1)} [\gamma_0(c^*)^{\gamma_1} - \gamma_1(c^*)^{\gamma_0}] + \frac{K}{c^*}$$

with $c^* = a^*/b^*$ being the unique real root bigger than 1 to the following equation:

$$f(x) = \frac{1}{\gamma_0 - \gamma_1} [\gamma_1(1 - \gamma_0)x^{\gamma_0} - \gamma_0(1 - \gamma_1)x^{\gamma_1}] - \frac{K}{l - K} = 0.$$

(ii) When $r < \max(0, \mu)$, $V^*(x, l) = \infty$.

Note that $f(x)$ is an increasing function with $f(1) < 0$ and $f(\infty) > 0$, therefore $c^* > 1$ is unique when $r > \max(0, \mu)$ (hence $\gamma_0 > 1 > 0 > \gamma_1$). That $b^* < l$ is due to the fact that $\gamma_0 y^{\gamma_1} - \gamma_1 y^{\gamma_0}$ is a decreasing function and $c^* > 1$. It is also easy to check that $V(x, l)$ and a^* specified in the theorem satisfy $V(x, l) > l - K$ and $a^* > l$ when $r > \mu$.

In particular, when $r > 0$ and $r \searrow \mu$, then $\gamma_0 \searrow 1$, hence $a^* \rightarrow \infty$ and $c^* \rightarrow \infty$.

Corollary 1. When $r = \mu > 0$, $V^*(x, l) = V(x, l) \geq l - K$ such that

$$V(x, l) = \begin{cases} l - K, & x \leq l^*, \\ (l - K) \left(\frac{\gamma_1}{\gamma_1 - \gamma_0} \left(\frac{x}{l}\right)^{\gamma_0} - \frac{\gamma_0}{\gamma_1 - \gamma_0} \left(\frac{x}{l}\right)^{\gamma_1} \right), & x > l^*. \end{cases}$$

It is easy to see that, when $r = 0 > \mu$, $V^*(x, l) = l - K$.

1.3. Options for risk-averse investors

Consider the following optimal stopping problem:

$$V^*(x, l) = \sup_{0 \leq \tau \leq \infty} E[e^{-r\tau} Y_\tau], \tag{8}$$

where $Y_t = \max(l, X_t)$ for a fixed $l > 0$, $x = X_0$, and r is a positive constant.

This problem arises as a direct consequence of the study of a naive, risk-averse investor who holds certain stocks and wishes that, even when the market falls down steeply against him, his risk is partly removed by getting at least some promised price. This gives rise to ‘riskless’ options which allows the owner to choose an exercise date, represented by the stopping time τ , and then pays the owner either l or the stock price X_τ . The owner of the option will naturally seek an exercise strategy that will maximize the present discounted value of his future reward.

Using similar techniques as before, we have the following result.

Theorem 3. Assume that $X = (X_t)$ as in (1).

(i) The optimal stopping time τ_* exists when $r > \max(0, \mu)$ and has the form

$$\tau_* = \inf\{t \geq 0 \mid X_t \notin (al, bl)\},$$

where $0 < a < 1 < b$ are such that

$$\begin{aligned} a &= \frac{\gamma_1}{\gamma_1 - 1} \left(\frac{\gamma_1(\gamma_0 - 1)}{\gamma_0(\gamma_1 - 1)} \right)^{(1-\gamma_0)/(\gamma_0-\gamma_1)}, \\ b &= \frac{\gamma_0}{\gamma_0 - 1} \left(\frac{\gamma_0(\gamma_1 - 1)}{\gamma_1(\gamma_0 - 1)} \right)^{\gamma_1/(\gamma_0-\gamma_1)}. \end{aligned} \tag{9}$$

(ii) When $r < \max(0, \mu)$, $P(\tau_* < \infty) = 0$.

Theorem 4. (i) When $r < \max(0, \mu)$, $V^*(x, l) = \infty$.

(ii) When $r > \max(0, \mu)$, then $V^*(x, l) = V(x, l)$, where

$$V(x, l) = \begin{cases} l, & x \leq al, \\ \frac{l}{\gamma_0 - \gamma_1} \left[\gamma_0 \left(\frac{x}{al}\right)^{\gamma_1} - \gamma_1 \left(\frac{x}{al}\right)^{\gamma_0} \right], & al < x < bl, \\ x, & bl \leq x. \end{cases} \tag{10}$$

Note that when $r = \mu > 0$, Theorem 4 remains valid except that $b \rightarrow \infty$, because $\gamma_0 = 1$ (hence $a = 1$). Therefore, the value function $V^*(x, l)$ is finite such that $V^*(x, l) = l$ when $x < l$;

$$V^*(x, l) = l \left(\left(\frac{x}{al}\right)^{\gamma_1} \left(\frac{\gamma_0}{\gamma_0 - \gamma_1}\right) - \left(\frac{x}{al}\right)^{\gamma_0} \left(\frac{\gamma_1}{\gamma_0 - \gamma_1}\right) \right)$$

when $x \in (l, \infty)$. Notice that the value function is discontinuous at $r = \mu > 0$. This case is specially interesting when considering an ‘arbitrage-free’ market where $r = \mu$ is imposed. In this case, even though the ‘fair price’ $V(x, l)$ is finite, those who crave better returns than l in finite time will get only an ε -optimal value. Note that when $r = 0 > \mu$, $V^*(x, l) = l$.

Despite the simplicity of the answer, we are not aware of any prior explicit solution to this problem. As was mentioned earlier, the derivation is based on the principle of smooth fit. Our solution has some new features that are not present in other stopping time problems. For instance, the boundary is not a constant as in perpetual American put and Russian options [10], [8]. Indeed, we feel that this is probably one of the nicest and simplest examples with both explicit solutions and nontrivial boundaries.

2. Proofs of the theorems

The proofs of the theorems are via martingale theory. Since the proof of Theorem 2 is similar to that of Theorem 3 and 4, we will present in detail only the proofs of Theorems 3, 4 and 1.

2.1. Proof of Theorems 3 and 4

Our proof is based on the following sequence of lemmas.

Lemma 1. For a, b given by (9), $0 < a < 1 < b$.

Proof. We have

$$b > 1 \iff \frac{\gamma_0}{\gamma_0 - 1} > \left(\frac{\gamma_0(\gamma_1 - 1)}{\gamma_1(\gamma_0 - 1)}\right)^{\gamma_1/(\gamma_1 - \gamma_0)}.$$

Recalling that $\gamma_0 > 1 > 0 > \gamma_1$, we have $0 < \gamma_1/(\gamma_1 - \gamma_0) < 1$, $(\gamma_1 - 1)/\gamma_1 > 1$, and $\gamma_0/(\gamma_0 - 1) > 1$, and therefore,

$$\left(\frac{\gamma_0(\gamma_1 - 1)}{\gamma_1(\gamma_0 - 1)}\right)^{\gamma_1/(\gamma_1 - \gamma_0)} < \left(\frac{\gamma_0}{\gamma_0 - 1}\right)^{\gamma_1/(\gamma_1 - \gamma_0)} < \frac{\gamma_0}{\gamma_0 - 1}.$$

Lemma 2. When $r > \max(0, \mu)$, $\sup_t \{e^{-rt} Y_t\}$ is integrable, that is, $E[\sup_t \{e^{-rt} Y_t\}] < \infty$, where $Y_t = \max(l, X_t)$.

Proof. See the proof of Equation (2.19) of [10].

Lemma 3. *The function $V(x, l)$ is C^1 on the boundaries $x = al$, $x = bl$ and is C^2 on the continuation region $x \in (al, bl)$, such that $V_{xx} \geq 0$, $V_x \geq 0$ and*

$$\begin{aligned} rV &= x\mu V' + \frac{1}{2}x^2\sigma^2 V'', & x \in (al, bl), & \quad (11) \\ V(x, l) &\geq l, & V(al, l) &= l, & V(bl, l) &= bl, \\ \frac{\partial V}{\partial x} \Big|_{x=al} &= 0, & \frac{\partial V}{\partial x} \Big|_{x=bl} &= 1. \end{aligned}$$

Proof. Direct calculations.

Now we proceed with the proof of Theorem 4.

Proof of Theorem 4. (i) is obvious. For (ii), let $Z_t = e^{-rt}V(X_t, l)$. We claim that Z_t is a supermartingale. Indeed, when $X_t \leq al < l$, we have

$$dZ_t = de^{-rt}l = -r e^{-rt}l dt \leq 0.$$

Moreover, in the region $X_t \in (al, bl)$,

$$\begin{aligned} dZ_t &= e^{-rt} [V_x(X_t, l) dX_t + \frac{1}{2}V_{xx}(X_t, l)(dX_t)^2 - rV(X_t, l) dt] \\ &= V_x(X_t, l)X_t\sigma dW_t, \end{aligned}$$

using (11). Therefore, Z_t is a local positive martingale when $X_t \in (al, bl)$.

In the region $X_t \geq bl$,

$$dZ_t = de^{-rt}X_t = e^{-rt}[(-r + \mu)X_t dt + \sigma dW_t].$$

Therefore, Z_t is a positive supermartingale, i.e.

$$E_{x,l} dZ_t \leq 0.$$

Consequently, for any stopping time τ , we can write

$$E_{x,l} e^{-r\tau} Y_\tau \leq E e^{-r\tau} V(X_\tau, l) = E_{x,l} Z_\tau \leq E_{x,l} Z_0 = V(x, l).$$

If we take the supremum over all such τ , we obtain for all x and fixed l ,

$$V^*(x, l) \leq V(x, l).$$

To prove the converse, define $\tilde{\tau}$ to be the first t for which $X_t \notin (al, bl)$. Then it is clear that $P(\tilde{\tau} < \infty) = 1$, due to the independent increments of Brownian motion.

Since $Z_t = e^{-rt}V(X_t, l) \leq CY_t$ from (10) and Lemma 2, we have

$$E \left[\sup_t Z_t \right] < \infty.$$

Applying the optional stopping theorem (cf. [9, p. 64]), we have, for any stopping time $\tilde{\tau}$,

$$E_{x,l} Z_{\tilde{\tau}} = E_{x,l} Z_0.$$

Taking the supremum of all stopping times, we get

$$V^*(x, l) \geq V(x, l).$$

This completes the proof of Theorem 4.

Surprisingly, when $l = 0$, the optimal stopping problem given by (8) completely degenerates.

Corollary 2. *When $l = 0$, (8) has a degenerate solution: when $r > \max(0, \mu)$, $V(x, l) \equiv x$.*

Proof. Noticing that $Z_t = e^{-rt}X_t$ is a supermartingale, the proof is immediate.

2.2. Proof of Theorem 1

To prove Theorem 1, we first need to prove the existence of $g(s)$. It is derived via the linearization theorem (interested readers are referred to [1] and [6] for some basic facts and techniques of ODEs).

Lemma 4. *There exists a unique $g(s)$ for (2) such that $0 \leq g(s) < as$, $g(K) = 0$, and $g(s)/s$ is asymptotic to a when $s \rightarrow \infty$.*

Proof. From (2), we see that around the neighborhood of $s = K$, $g(s)$ is $O(s - K)^{1/\gamma_0}$. Now, define $h(s) = g(s)/s$ and let

$$F(h) = \frac{1}{-\gamma_0\gamma_1} \frac{\gamma_0 h^{\gamma_0 - \gamma_1} - \gamma_1}{1 - h^{\gamma_0 - \gamma_1}}.$$

Then

$$h'(s) = \left(F(h) - \frac{s - K}{s} \right) \frac{h}{s - K}.$$

We see that

1. $F(h)$: for $h \geq 0$, we have $F(0) = 1/\gamma_0 \in (0, 1)$, $F(1^-) = +\infty$, and F is increasing for $h \in (0, 1)$. Let $a \in (0, 1)$ be such that $F(a) = 1$.
2. $l(s) = (s - K)/s$: for $s > K$, l is increasing, with $l(K) = 0$ and $l(\infty) = 1$.
3. Consider the (s, F) plane $s > K$ and $F > 1/\gamma_0$: except for a change of scale, as $F(h)$ is increasing, it corresponds to the plane (s, h) with $h \in (0, 1)$.
4. When $F > l$, $h' > 0$ and the solution $h(s)$ is increasing. On $F = l$, we have $h' = 0$ and $h(s)$ is a critical point. For $F < l$, $h(s)$ is decreasing.
5. On $h = 0$ (or $F = 1/\gamma_0$), the beginning of the solution is parallel to the s -axis. For $h = a$ (or $F = 1$), the solution increases (as $l < 1$). For $h = 1$ (or $F = +\infty$), the solution goes (locally) vertically.

Now we see that there are two types of solutions—those which go monotonically to infinity (starting above $F = 1$ or crossing it) and those with a unique critical point (a maximum) always going below $F = 1$ (that is, $h = a$) and crossing l . Therefore, there must be at least one separating solution which increases above l but below $F = 1$.

Furthermore, in the neighborhood of infinity, define $H(t) = h(1/t)$, let τ be such that $d\tau/dt = -1/(t(1 - Kt))$, and define $H = \hat{a}$ when $h = a$; then the following system has the same trajectories:

$$\begin{cases} \frac{dt}{d\tau} &= -t(1 - Kt), \\ \frac{dH}{d\tau} &= H(F(H) - 1 + Kt). \end{cases}$$

Since H is nonnegative, this system has three critical points $(0, 0)$, $(1/K, 0)$, $(0, \hat{a})$ (that is, $F = 1$). The linearization matrix is given by

$$\begin{bmatrix} -1 + 2Kt & 0 \\ KH & F + F'H - 1 + Kt \end{bmatrix}.$$

Hence, at $(0, 0)$ we get

$$\begin{bmatrix} -1 & 0 \\ 0 & 1/\gamma_0 - 1 \end{bmatrix},$$

that is, a sink as $\gamma_0 > 1$. At $(1/K, 0)$ we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1/\gamma_0 \end{bmatrix},$$

that is, a source and at $(0, \hat{a})$ we get

$$\begin{bmatrix} -1 & 0 \\ K\hat{a} & F'\hat{a} \end{bmatrix},$$

that is, a saddle as $F' > 0$ and $\hat{a} > 0$. Therefore, all the trajectories which are not escaping to $+\infty$ (along the separatrix $t = 0$) are attracted by $(0, 0)$ except for the second separatrix of the saddle $(0, \hat{a})$.

In conclusion, we see that there is a unique solution $g(s)$ asymptotic to the line $g = as$. The rest either cross the diagonal $g = s$ and go back (not giving a functional solution $g(s)$, but rather $s(g)$) or converge to 0 as $s \rightarrow \infty$.

Now define $Y_t = e^{-rt}V(X_t, S_t)$. We can check that Y_t is a positive supermartingale. Moreover, we have the next lemma.

Lemma 5. *We have*

$$\mathbf{E} \sup_{0 \leq t < \infty} Y_t < \infty.$$

Proof. Note that

$$Y_t = e^{-rt}V(X_t, S_t) \leq e^{-rt}V(S_t, S_t).$$

It suffices to show that

$$V(S_t, S_t) = \frac{S_t - K}{\gamma_0 - \gamma_1} \left(\gamma_0 \left(\frac{S_t}{g(S_t)} \right)^{\gamma_1} - \gamma_1 \left(\frac{S_t}{g(S_t)} \right)^{\gamma_0} \right) \leq KS_t. \quad (12)$$

Given (12), Lemma 5 follows immediately from the fact that when $r > \mu$, $\sup_t \exp(-rt)S_t$ is integrable (cf. Equation (2.19) of [10]).

Now define

$$d(y) = \gamma_0 \left(\frac{y}{g(y)} \right)^{\gamma_1} - \gamma_1 \left(\frac{y}{g(y)} \right)^{\gamma_0}.$$

Recall that $g(y) = ay$ as $y \rightarrow \infty$, and, for any $\varepsilon > 0$, there exists an $M(\varepsilon)$ such that when $y \geq M$, $|g(y) - ay| \leq a\varepsilon$. Therefore $d(y)$ is bounded both when $y < M$ and $y > M$, which validates (12).

Now following the same procedure as in Section 2.1, the proof can be completed.

2.3. Some remarks on Theorem 1

Remark 1. It is worth pointing out that the discussion of Theorem 1 goes through without much difficulty if the payoff function $(s - K)^+$ is replaced by a function $c(s)$ with the following properties:

- (i) $c(s) = 0$, when $s < K$;
- (ii) $\lim_{s \rightarrow \infty} c(s)/s$ exists and is equal to some positive constant;

- (iii) for $s > K$, $c(s)$ has a derivative strictly bounded away from zero;
 (iv) $\tilde{l}(s) = c(s)/(sc'(s))$ is increasing to a finite limit which is bigger than $F(0)$.

Under these assumptions, the equation for $h(s) = g(s)/s$ is

$$h'(s) = h(s)c(s)(F(h) - \tilde{l}(s))/c'(s).$$

Repeating the same analysis as that for Theorem 1 leads to solutions for the following optimal stopping problem:

$$V^*(x, s) = \sup_{0 \leq \tau \leq \infty} E_{x,s}[e^{-r\tau} c(S_\tau)]. \quad (13)$$

The corresponding value of a is different, though. It is given by $F(a) = \lim_{s \rightarrow \infty} \tilde{l}(s)$. (For simplicity, Theorem 1 illustrates the analysis via a concrete example of such a function $c(s) = (s - K)^+$.)

In short, we have the following result.

Theorem 5. (i) Assuming properties (i)–(iv), the value function $V^*(x, s)$ in (13) is finite if and only if $r > \max(0, \mu)$.

(ii) When $r > \max(0, \mu)$,

$$V^*(x, s) = \begin{cases} c(s), & x \leq g(s), s > K, \\ \frac{c(s)}{\gamma_0 - \gamma_1} \left(\gamma_0 \left(\frac{x}{g(s)} \right)^{\gamma_1} - \gamma_1 \left(\frac{x}{g(s)} \right)^{\gamma_0} \right), & s > x \geq g(s), s > K, \\ A \frac{\gamma_1}{\gamma_1 - \gamma_0} x^{\gamma_0}, & s < K, x < s, \end{cases}$$

where $0 \leq g(s) < as$ for all $s > K$ and $g(s)$ is the (unique) solution to the differential equation

$$c'(s) \left[\gamma_0 \left(\frac{s}{g(s)} \right)^{\gamma_1} - \gamma_1 \left(\frac{s}{g(s)} \right)^{\gamma_0} \right] = -\gamma_0 \gamma_1 c(s) \frac{g'(s)}{g(s)} \left[\left(\frac{s}{g(s)} \right)^{\gamma_0} - \left(\frac{s}{g(s)} \right)^{\gamma_1} \right],$$

such that when $s \rightarrow \infty$, $g(s)/s$ is asymptotic to a , that is, $\lim_{s \rightarrow \infty} g(s)/s = a$ and $0 \neq A = \lim_{s \rightarrow K^+} c(s)/g(s)^{\gamma_0}$. Here, $\gamma_0 > 1 > 0 > \gamma_1$ are the solutions of

$$r = \mu\gamma + \frac{1}{2}\sigma^2\gamma(\gamma - 1),$$

and a satisfies

$$F(a) = \lim_{s \rightarrow \infty} \frac{c(s)}{sc'(s)},$$

where

$$F(x) = \frac{1}{-\gamma_0\gamma_1} \frac{\gamma_0 x^{\gamma_0 - \gamma_1} - \gamma_1}{1 - x^{\gamma_0 - \gamma_1}}.$$

The optimal stopping time τ^* is

$$\tau^* = \inf\{t \geq 0 \mid X_t = g(S_t), S_t > K\},$$

starting from $X_0 = x > 0$, $S_0 = s$, $x > g(s)$.

Remark 2. It is often the case in explicit solutions of optimal stopping problems that smooth fit uniquely identifies the boundary, and vice versa (cf. [4], [5], [7], [8], [12] and the references therein). An interesting feature of the solution to the perpetual lookback American option is the fact that smooth fit itself is not enough. Rather, we need an additional condition to ensure proper growth of the value function in order to carry out the martingale argument. This is why the condition on the asymptotic behavior of $g(s)$ arises.

Remark 3. In [10], a ‘Russian option’ was defined and a similar, albeit simpler, optimal stopping time problem was studied. Although their problem corresponds to a special case of our problem with $K = 0$, both the payoff functions and the solution structure are different. In particular, unlike the Russian option case, in which the free boundary is a straight line $x = as$, our solution is more sophisticated, yet explicit. It is interesting to see that as s approaches ∞ , the free boundary $x = g(s)$ in our problem goes asymptotically to $x = as$, with the same value for a as in [7]. Indeed, if we let $K = 0$, then (6) plus the boundary condition $g(0) = 0$ yields the a as given by (7).

Finally, although it is possible to provide a (simpler) derivation of the Russian option problem based on the observation that X_t/S_t is Markovian (cf. [11]), the free boundary in our problem $x = g(s)$ seems totally unconnected to the fact that X_t/S_t is Markovian. It further confirms our understanding that the structure of their solution is also intrinsically related to the linearity of the payoff function (because $K = 0$), as was pointed out in [3].

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