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SOME RISK MANAGEMENT PROBLEMS FOR FIRMS WITH INTERNAL COMPETITION AND DEBT

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Abstract

Consider an optimization problem for a company with the following parameters: a constant liability payment rate (δ), an average return (μ) and a risk (σ) proportional to the size of the business unit, and an internal competition factor (α) between different units. The goal is to maximize the expected present value of the total dividend distributions, via controls (U_t, Z_t), where U_t is the size of the business unit and Z_t is the total dividend payoff up to time t. We formulate this as a stochastic control problem for a diffusion process X_t and derive an explicit solution by solving the corresponding Hamilton–Jacobi–Bellman equation. The resulting optimal control policy involves a mixture of a nonlinear control for U_t and a singular control for Z_t. The optimal strategies are different for the cases $\delta > 0$ and $\delta = 0$. When $\delta > 0$, it is optimal to play bold: the initial optimal investment size should be proportional to the debt rate δ. Under this optimal rule, however, the probability of bankruptcy in finite time is 1. When $\delta = 0$, i.e. when the company is free of debt, the probability of going broke in finite time reduces to 0. Moreover, when $\delta = 0$, the value function is singular at $X_0 = 0$. Our analytical result shows considerable consistency with daily business practices. For instance, it shows that 'too many people is counter-productive'. In fact, the maximal optimal size of the business unit should be inversely proportional to α. This eliminates the redundant and simplistic technical assumption of a known uniform upper bound on the size of the firm.

Keywords: Hamilton–Jacobi equation; singular control; nonlinear control; dividend optimization; internal competition

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1. Introduction

The following stochastic control problem and its variations were considered in [12] and [14]:

$$V^*(x) = \sup_{U_t, Z_t} \mathrm{E}_x \int_0^\tau \mathrm{e}^{-rt} \, \mathrm{d}Z_t, \tag{1}$$

where

$$\mathrm{d}X_t = \mu U_t \, \mathrm{d}t + \sigma U_t \, \mathrm{d}W_t - \mathrm{d}Z_t.$$

Here, μ, σ, r are positive constants, W_t is a one-dimensional Brownian motion on a probability space $(\mathcal{F}, \mathrm{P}, \mathcal{F}_t)$, U_t and Z_t are two adapted control processes with respect to $\mathcal{F}_t$, $\tau = \inf\{t > 0 \mid X_t = 0\}$ and $X_0 = x > 0$.

The initial motivation behind the problem in (1) was the following. For a firm, let U_t be the size of the business unit at time t and let Z_t be the dividend payoff up to time t. Then (1)

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represents a profit/dividend optimization problem for the firm under an appropriate hiring policy and a careful dividend taking. Using an intensive and delicate analysis, [12] and [14] provide explicit solutions and linear optimal (Markov) control policies under different constraints. However, all the solutions relied on a crucial technical assumption: *a known uniform upper bound M for U_t.*

Stochastic control problems of this type arise in other fields as well, especially in insurance [1], [2], [3], [4], [6], [10], [16]. Choulli *et al.* [6] and Taksar and Zhou [16] consider risk-control problems for insurance companies which employ a 're-insurance practice' to divert part of their risk, where $1 - U_t$ signifies the re-insurance fraction. Their goal was to maximize the expected return of the total dividend distributions. Again, their solutions (with linear optimal controls) rely on the boundedness of U_t. Although it was natural for [6] to assume that $U_t \leq M < 1$, the uniform boundedness assumption on the size of a company in (1) is at best simplistic, because hardly any two companies will share the same view. Recently, Asmussen *et al.* [3] considered a more general stochastic control problem on a diffusion process with more complex drift and diffusion terms. Their 'parameterization method', however, relies on assuming a monotonicity relation between the drift and the diffusion functions.

1.1. Our problem

In this paper, we consider the optimization problem of (1) when the drift term is no longer linear and there is no one-to-one correspondence between the drift and the diffusion terms. As in [12], we assume that μ is the expected *net* profit per unit generated with risk factor σ. Thus, the company with staff size U_t will have a potential net profit μU_t and a risk σU_t. In addition, we incorporate two important scenarios from the real world: (i) an *internal competition factor* inside the firm; (ii) *a liability* that a firm has to face—the company has to pay debt at a (constant) rate.

We formulate this mathematically as follows. Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathrm{P})$ and a standard Wiener process W_t, we assume that the capital of the company at time t is X_t, with $X_0 = x \geq 0$, and that the company's profit taking is represented by an adapted process $Z = (Z_t)$. Let

$$\tau = \inf\{t > 0 \mid X_t = 0\}$$

be the time the company declares bankruptcy, and $\mathrm{d}X_t = \mathrm{d}Z_t = 0$ when $t > \tau$. Then, the dynamics of the process X_t can be described by the equation

$$\mathrm{d}X_t = U_t(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t) - \alpha U_t^2\,\mathrm{d}t - \mathrm{d}Z_t - \delta\,\mathrm{d}t. \tag{2}$$

Here, $U_t > 0$ is an adapted (with respect to $\mathcal{F}_t$) bounded process that represents the size of the firm at time t, α is the internal 'competition factor' between different units inside the firm, $\delta \geq 0$ is the debt rate the company faces, $\mu > 0$ is the average profit each unit generates, $\sigma > 0$ is the 'uncertainty' (ups and downs) of each unit, and $\mathrm{d}Z_t \geq 0$ is such that Z_t is a nonnegative, nondecreasing adapted process representing the total profit taken up to time t, with $Z_0 = z$.

We restrict our attention to admissible controls, denoted $\mathcal{B}$, for U_t and Z_t. In other words, Z_t and U_t are all right-continuous processes with left limits.

Using this model, our goal is to maximize the time-discounted total expected profit, via proper choices of (U_t, Z_t), i.e. to find the value of

$$V^*(x) = \sup_{Z_t, U_t} \mathrm{E}_x \int_0^\tau \mathrm{e}^{-rt}\,\mathrm{d}Z_t, \tag{3}$$

where τ is the time of bankruptcy and $r > 0$ is the discounted factor.

1.2. Our results

From standard control theory (cf. [7]), we know that answers to the above question usually involve solving certain Hamilton–Jacobi–Bellman-type (HJB) equations. Although solving HJB equations explicitly can be formidable in general, we succeed in our case via a delicate analysis. It turns out that the optimal control policy is a mixture of a nonlinear regular control for U_t and a singular control for Z_t.

Our mathematical results provide some interesting insights that are consistent with daily business practice.

(i) We show that 'too many people may be counter-productive'. When the debt rate is not high, i.e. when $\mu > (4\alpha\delta)^{1/2}$, the maximal size of the company is $2\mu/\alpha$. This comes as no surprise—the more efficient a company, the smaller the factor α, the bigger the company can afford to grow. When the debt rate is high, i.e. when $\mu \leq (4\alpha\delta)^{1/2}$, it is optimal to 'take the money and run'.

This result also eliminates the redundant assumption in [14] where a uniform upper bound M for U_t is needed for derivation of a finite and unique value function.

(ii) We prove that when the debt rate is nonzero, and when it is within a reasonable range, i.e. $\mu > (4\alpha\delta)^{1/2}$, it is optimal to play 'cautiously bold'. More concretely, the optimal (Markov) control policy $U(x)$ is such that $U(0) = 2\delta/\mu$: the higher the debt, the bigger (!) the risk one needs to take. On the other hand, the higher the return of each unit, the lesser the risk to start a business.

(iii) It is never optimal for the company to take profit before it reaches its full gear—do not take profit until the company grows to the maximal optimal size, after which profit-taking should be immediate.

(iv) Interestingly, the optimal strategies when $\delta > 0$ and $\delta = 0$ are different. When $\delta > 0$, although it is optimal to play bold, under this optimal strategy, the probability of bankruptcy in finite time is 1. When $\delta = 0$, the probability of going broke in finite time is 0. This is not surprising. If a company is free of debt, it can manipulate carefully near the edge of bankruptcy by playing cautiously to prolong its life so long as it has potential positive profit. If a company has a constant debt rate, it has little room to spare when it is near bankruptcy.

As a by-product, we rectify the imprecise statement concerning ruin probability in [12] and [14], which should be 0 (in most cases), instead of being 1. Philosophically it is intriguing: in [12] and [14], a Bachelier's Brownian motion model was adopted in favour of the well-known Black–Scholes' geometric Brownian motion model. Nevertheless, in order to improve the survival chance, under the optimal risk control model, capital assets do fluctuate as a geometric Brownian motion in the neighbourhood of bankruptcy. This is an interesting resemblance between the evolution of a company's assets and stock-price fluctuations.

2. Problems and solutions

Throughout the paper, $r > 0$ unless specified otherwise.

Assuming (2), with a pair of controls $\pi = (U_t^\pi, Z_t^\pi)$ with (U_t^π, Z_t^π) all in $\mathscr{B}$, denote its performance index by

$$J_x(\pi) = \mathrm{E}_{x,\pi} \int_0^\tau \mathrm{e}^{-rt}\, \mathrm{d}Z_t.$$

The objective is to find

$$V^*(x) = \sup_\pi J_x(\pi),$$

and the admission control π^* such that

$$V^*(x) = J_x(\pi^*) = \sup_{\pi} J_x(\pi^*).$$

The function $V^*(x)$ is called the *(optimal) value function*, and π^* is called the *optimal policy*.

First, from standard control theory, we have

Proposition 1. *The function $V^*(x)$ is concave and nondecreasing in x with $V^*(0) = 0$.*

Furthermore, if the value function $V^*(x)$ is twice differentiable, then the dynamic programming principle suggests that $V^*(x)$ is the solution to the following well-known HJB equation:

$$\max_{U,Z}[(1 - V_x), -rV(x) + (\mu U(x) - \alpha U^2(x) - \delta)V_x + \tfrac{1}{2}\sigma^2 U(x)^2 V_{xx}] = 0. \tag{4}$$

(Interested readers are referred to [3] for the proofs.)

As was pointed out in [15], solving the HJB equation does not guarantee the optimality of the solution; we do not even know *a priori* if the solution is unique. Nevertheless, we begin our analysis by trying to solve the HJB equation (4) heuristically. Then, by proving a modified version of the well-known verification theorem, we show that any concave solution to (4) majorizes the performance functional for any policy π. Consequently, the solution yields the optimal policy and is unique.

The following sections are devoted to the delicate analysis for solving the HJB equations. Although the methods of solving the equations are similar for $\delta = 0$ and $\delta > 0$, the features of the solutions and optimal strategies are different. For the sake of clarity, we investigate these two cases separately.

2.1. The case $\delta > 0$

In order to solve (4), we begin with some heuristics. Starting at time $t = 0$, $X_t = x > 0$ and assuming the control variable $U(\cdot)$ is a feedback control, i.e. it is a function of x, we should choose to allow the capital as well as the size of the company to grow as long as the capital X_t enjoys a nonnegative drift. However, because of the extra term $-\delta\, dt$, the diffusion process may always have a negative drift. If $\mu \le (4\alpha\delta)^{1/2}$, then $f(y) = -\alpha y^2 + \mu y - \delta$ is always nonpositive, and so is the drift. In this case, the company has a mounting debt rate that outpaces the potential profit rate. Intuition tells us that we had better 'take the money and run'. In other words, we have the following result.

Theorem 1. *We have $V^*(x) \equiv x$ if and only if $\mu \le (4\alpha\delta)^{1/2}$.*

Proof. Recall that $V^*(x)$ is a concave, nondecreasing function with $V^*(0) = 0$ and, therefore, $V^*_{xx} \le 0$, $V^*_x \ge 0$. Moreover, when $\mu \le (4\alpha\delta)^{1/2}$, $f(y) = \mu y - \alpha y^2 - \delta < 0$ for all y; thus,

$$\max_U(\tfrac{1}{2}\sigma^2 U^2 V^*_{xx}(x) + (\mu U - \alpha U^2 - \delta)V^*_x(x) - rV^*(x)) \le -rV^*(x) < 0.$$

Hence, the HJB equation (4) is satisfied if and only if $V^*_x(x) = 1$, for any x. The initial condition $V^*(0) = 0$ yields $V^*(x) = x$, that is, $V^*(x) \equiv x$. This is achievable—the optimal policy is to stop immediately and take profit x. (To check the optimality, we can also directly apply Theorem 2.2 of [6].)

From now on, we assume that

$$\mu \geq \sqrt{4\alpha\delta}.$$

Continuing our heuristic, we speculate that the optimal $U(x)$ should be a nondecreasing nonnegative function until a threshold x^*. Noticing that the maximum of $y(\mu - \alpha y)$ is $\mu^2/(4\alpha)$, it is reasonable to believe that at this point $x = x^*$, $U(x) = \mu/(2\alpha)$, where the capital process X_t reaches its largest possible drift. When $x > x^*$, we can start taking profit, i.e. we should push x as quickly as possible back to x^* in order to keep this drift at this maximal level. In fact, after careful analysis, we find that profit taking should take place only after (and *immediately* after!) $x \geq x^*$.

Analytically, we are looking for a smooth function $\bar{V}(x)$ such that

$$\bar{V}(x) = \begin{cases} V_1(x) & x \leq x^*, \mathrm{d}Z = 0, \\ V_2(x) & x > x^*, \mathrm{d}Z > 0, U(x) \equiv \mu/(2\alpha), \end{cases}$$

with

$$x^* = \inf\left\{ x > 0 \,\middle|\, U(x^*) = \frac{\mu}{2\alpha} \right\},$$

for some control function $U(x)$.

Now we need to find this control $U(x)$ when $x < x^*$, together with x^*. To this end, taking the supremum over all controls $U(x)$ from (4) leads to

$$(\mu - 2\alpha U(x))\bar{V}_x + \sigma^2 U(x)\bar{V}_{xx} = 0. \tag{5}$$

On the other hand, assuming that $\bar{V}'(x) \geq 1$, (4) yields

$$-r\bar{V} + (\mu U(x) - \alpha U^2(x) - \delta)\bar{V}_x + \tfrac{1}{2}\sigma^2 U^2(x)\bar{V}_{xx} = 0. \tag{6}$$

Hence, the following relations hold for $V_1(x)$ and $U(x)$ when $x \leq x^*$:

$$\begin{aligned} -rV_1 + (\mu U(x) - \alpha U^2(x) - \delta)V_{1x} + \tfrac{1}{2}\sigma^2 U^2(x)V_{1xx} &= 0, \\ (\mu - 2\alpha U(x))V_{1x} + \sigma^2 U(x)V_{1xx} &= 0. \end{aligned} \tag{7}$$

Solving (7), we arrive at

$$(\mu U - 2\delta)V_{1x} = 2rV_1. \tag{8}$$

Now, taking the derivative of (8), we get

$$(\mu U'(x) - 2r)V_{1x} = (2\delta - \mu U(x))V_{1xx}. \tag{9}$$

Thus, if we assume that

$$V_{1x}(x) \neq 0, \qquad V_{1xx}(x) \neq 0, \tag{10}$$

for $x < x^*$, then, in view of (9) and (6), we can derive the equation for $U(x)$ when $x < x^*$:

$$\sigma^2(2r - \mu U'(x))U(x) = (2\delta - \mu U(x))(\mu - 2\alpha U(x)). \tag{11}$$

Moreover, (8), suggests that $U(0) = 2\delta/\mu$, since we require that $V_1(0) = V^*(0) = 0$.

Now, $U(x)$ can be determined uniquely, from standard ODE theory.

Lemma 1. *There exists a unique solution $U(x)$ such that*

$$\begin{aligned}\sigma^2(2r - \mu U'(x))U(x) &= (\mu - 2\alpha U(x))(2\delta - \mu U(x)),\\ U|_{x=0} &= \frac{2\delta}{\mu}.\end{aligned} \tag{12}$$

In fact, we can solve this equation in a closed form. After some calculations, we see that

$$[A - U(x)]^{A/(A-B)}[U(x) - B]^{-B/(A-B)} = K\exp\left(-\frac{2\alpha}{\sigma^2}x\right),$$

where

$$A = \frac{2r\sigma^2 + \mu^2 + 4\alpha\delta + \sqrt{(2r\sigma^2 + \mu^2 + 4\alpha\delta)^2 - 16\alpha\mu^2\delta}}{4\alpha\mu},$$

$$B = \frac{2r\sigma^2 + \mu^2 + 4\alpha\delta - \sqrt{(2r\sigma^2 + \mu^2 + 4\alpha\delta)^2 - 16\alpha\mu^2\delta}}{4\alpha\mu},$$

and

$$K = [A - U(0)]^{A/(A-B)}[U(0) - B]^{-B/(A-B)}.$$

Here, A, B are real positive numbers, since

$$(2r\sigma^2 + \mu^2 + 4\alpha\delta)^2 > (2r\sigma^2 + \mu^2 + 4\alpha\delta)^2 - 16\alpha\mu^2\delta = (2r\sigma^2 - \mu^2 + 4\alpha\delta)^2 + 8r\mu^2\sigma^2 > 0.$$

Moreover, it is easy to check that $A > U(0) = 2\delta/\mu > B$.

To continue our analysis, we need to prove the existence of $x^* > 0$ for which $U(x^*) = 2\alpha/\mu$ with $U(x)$ given by Lemma 1.

Lemma 2. *Assume that $U(x)$ is the solution to (12); then there exists an $x^* > 0$ such that $U(x^*) = 2\alpha/\mu$.*

Proof. Recall that for $x < x^*$, $U(x) < \mu/(2\alpha)$, $U(0) = 2\delta/\mu$. Since $\mu > (4\alpha\delta)^{1/2}$, we have $U^*(0) = 2\delta/\mu < 2\alpha/\mu$. Now, from (12), it is easy to check that $U(x)$ is a strictly increasing function before reaching the value of $\mu/(2\alpha)$. Therefore, there exists an x^* such that $U(x^*) = 2\alpha/\mu$.

Moreover, we have the next result.

Proposition 2. *When $\delta \neq 0$ and $U'(0) = 2r/\mu$, $U''(0) = r(\mu^2 - 4\alpha\delta)/(\mu\delta\sigma^2)$, for $U(x)$ given by Lemma 1.*

Proof. The first claim is obvious from (12), since $\mu/(2\alpha) > U(0) = 2\delta/\mu > 0$.

Furthermore, from (12), we see that

$$\lim_{x\to 0}\frac{2r - \mu U'(x)}{\mu U(x) - 2\delta} = \lim_{x\to 0}\frac{2\alpha U(x) - \mu}{U(x)\sigma^2}.$$

Therefore,

$$\lim_{x\to 0}\frac{-\mu U''(x)}{\mu U'(x)} = \frac{2\alpha U(0) - \mu}{U(0)\sigma^2} = \frac{4\alpha\delta - \mu^2}{2\delta\sigma^2},$$

from which we get

$$U''(0) = U'(0)\frac{\mu^2 - 4\alpha\delta}{2\delta\sigma^2} = \frac{r(\mu^2 - 4\alpha\delta)}{\mu\delta\sigma^2} > 0.$$

Now define

$$x^* = \inf\left\{x > 0 \,\middle|\, U(x) = \frac{\mu}{2\alpha}\right\}, \tag{13}$$

and

$$U^*(x) = \begin{cases} U(x) & x < x^*, \\ \dfrac{\mu}{2\alpha} & x > x^*, \end{cases} \tag{14}$$

with $U(x)$ being the solution to (12) and x^* given by (13) and Lemma 2.

Now, we are poised to decide the value functions $V_1(x)$ and $V_2(x)$. Recall that (9) and (11) imply that

$$V_{1xx}|_{x=x^*} = 0,$$

which suggests that x^* is the first point from which $V_{1x}(x)$ is nonincreasing. Since $y(u-\alpha y)$ is bounded by $\mu^2/(4\alpha)$, we see from (2) that $\lim_{x\to\infty} V_2(x) \le Bx$ for some constant B. Taking into account the concavity of the nondecreasing function $V_2(x)$ (from Proposition 1) and (2) and (3), we are led to the natural guess that for $x > x^*$,

$$V_2 = x + \eta.$$

In order to determine η and $V_1(x)$, we apply the 'principle of smooth fit'. Loosely speaking, 'smooth fit' dictates that a value function is, in general, smooth near the threshold boundary (cf. [5], [12]). In our case, it suggests that we should have

$$V_{1x}(x^*) = V_{2x}(x^*) = 1,$$
$$V_1(x^*) = V_2(x^*),$$

that is, $\eta = V_1(x^*) - x^*$.

Now we are in the position to solve for $V_1(x)$. If our heuristic is true, then $V_1(x)$ is decided by (8) and Lemma 1 if we have the next lemma.

Lemma 3. *There exists a unique concave function* $V_1(x),\ x \le x^*$, *with* $V_1(0) = 0$ *such that*

$$\begin{aligned} 2rV_1(x) &= (\mu U^*(x) - 2\delta)V_{1x}(x), \\ V_{1x}|_{x=x^*} &= 1. \end{aligned} \tag{15}$$

Moreover, $V_{1x} > 0$ *and* $V_{1xx} < 0$ *for* $x < x^*$.

Proof. The existence and the uniqueness of the solution are easily seen from the standard ODE theory, except for the point $x = 0$. It remains to show that for $V_1(x)$ satisfying (15), we have $V_1(0) = 0$ and that $V_1(x)$ is concave.

To show that $V_1(0) = 0$, using Proposition 2, we have

$$\lim_{x\to 0} \frac{xV_{1x}(x)}{V_1(x)} = \lim_{x\to 0} \frac{2rx}{U^*(x) - 2\delta} = \lim_{x\to 0} \frac{2r}{\mu U^*_x(0)} = 1.$$

Now, since $U^*_{xx}(0) > 0$, using Proposition 2, we can write, in the neighbourhood of $x = 0$,

$$U^*(x) = U^*(0) + U^*_x(0)x + \tfrac{1}{2}U^*_{xx}(0)x^2 + \cdots .$$

Therefore,

$$\begin{aligned}\int \frac{V_{1x}(x)}{V_1(x)}\,\mathrm{d}x &= \int \frac{2r}{\mu U^*(x) - 2\delta}\,\mathrm{d}x \\ &= \int \frac{2r}{\mu(U_x^*(0)x + \frac{1}{2}U_{xx}^*(0)x^2 + \cdots)}\,\mathrm{d}x \\ &= \int \frac{2r}{2rx + \frac{1}{2}\mu U_{xx}^*(0)x^2 + \cdots}\,\mathrm{d}x.\end{aligned}$$

Now, near the neighbourhood of $x = 0$, we can choose constants a and a' with $0 < a < U_{xx}^*(0)/(2U_x^*(0)) < a'$ such that

$$\int \frac{1}{x + a'x^2}\,\mathrm{d}x \le \int \frac{V_{1x}(x)}{V_1(x)}\,\mathrm{d}x \le \int \frac{1}{x + ax^2}\,\mathrm{d}x,$$

which yields

$$\exp(C)\frac{x(1 + a'x)}{a'} \le V_1(x) \le \exp(C)\frac{x(1 + ax)}{a},$$

with $C > 0$ being a constant uniquely determined by $V_{1x}(x^*)$. Taking $x \to 0$, we get $V_1(0) = 0$. Now we need to check that the solution is concave, i.e. that $V_{1xx} \le 0$. We know that when $x < x^*$, $2\delta/\mu \le U^*(x) \le 2\alpha/\mu$. Therefore, from (8) and (9), we see that $V_{1xx} \le 0$ and $V_1(x) > 0$ for $x < x^*$.

Lemma 3 also validates our previous assumption of (10). Moreover, from the proof of Lemma 3, the next result follows.

Proposition 3. *We have* $V_{1x}(0) < \infty$.

2.2. Optimal control policy and solutions

Theorem 2. *When* $\mu > (4\alpha\delta)^{1/2}$, $V^*(x) = \bar{V}(x)$, *with* $\bar{V}$ *given by*

$$\bar{V}(x) = \begin{cases} V_1(x) & x \le x^*, \\ x + V_1(x^*) - x^* & x > x^*, \end{cases}$$

where $V_1(x)$ *is given by Lemma 3 and* x^* *is defined by (13).*

Essentially, to prove Theorem 2 is to prove the optimality of the solution. This is generally called the 'verification theorem'. This proof is fairly standard, because of Proposition 3 (cf. [6]). To permit comparison with the $\delta = 0$ case, we provide below details of the proof. We start by checking some basic ingredients.

Lemma 4. *We have* $\bar{V}'(x) \ge 1$.

Proof. It suffices to check the case for $x < x^*$. Recall that x^* is the starting point from which $V_{1x}(x)$ is decreasing. Equation (5) reveals that for any $x < x^*$, $U^*(x) < 2\alpha/\mu$, and for $x = x^*$, $V_{1x} = 1$, and, therefore, $V_{1xx} \le 0$ for $x \le x^*$. Hence V_{1x} is decreasing for $x < x^*$ until $V_{1x}(x^*) = 1$.

Lemma 5. *We have*

$$-r\bar{V} + (\mu U^* - \alpha(U^*(x))^2)\bar{V}_x + \tfrac{1}{2}\sigma^2(U^*(x))^2\bar{V}_{xx} \le 0.$$

This is immediate, since $U^*(x)$ is the solution to (4).

Proof of Theorem 2. We have

$$\begin{aligned}\bar{V}(x) &= \mathrm{E}(\mathrm{e}^{-rT\wedge\tau}\bar{V}(X_{T\wedge\tau})) \\ &\quad - \mathrm{E}\int_0^{T\wedge\tau} \mathrm{e}^{-rt}(\tfrac{1}{2}\sigma^2U^2\bar{V}''(X_t) + (\mu U - \alpha U^2 - \delta)\bar{V}'(X_t) - rV(X_t))\,\mathrm{d}t \\ &\quad + \mathrm{E}\int_0^{T\wedge\tau} \mathrm{e}^{-rt}\bar{V}'(X_t)\,\mathrm{d}Z_t^{\mathrm{c}} - \mathrm{E}\sum_{0\le t\le T\wedge\tau} \mathrm{e}^{-rt}(\bar{V}(X_t) - \bar{V}(X_{t-})),\end{aligned}$$

where Z_t^{c} corresponds to the continuous part of Z_t. Since

$$\bar{V}(X_t) - \bar{V}(X_{t-}) = \bar{V}(X_{t-} - (Z_t - Z_{t-})) - \bar{V}(X_{t-}) \ge -(Z_t - Z_{t-}),$$

we have

$$\begin{aligned}\bar{V}(x) &= \mathrm{E}(\mathrm{e}^{-rT\wedge\tau}\bar{V}(X_{T\wedge\tau})) + \mathrm{E}\int_0^{T\wedge\tau} \mathrm{e}^{-rt}\,\mathrm{d}Z_t^{\mathrm{c}} + \mathrm{E}\sum_{0\le t\le T\wedge\tau} \mathrm{e}^{-rt}(Z_t - Z_{t-}) \\ &\ge \mathrm{E}(\mathrm{e}^{-rT\wedge\tau}\bar{V}(X_{T\wedge\tau})) + \mathrm{E}\int_0^{T\wedge\tau} \mathrm{e}^{-rt}\,\mathrm{d}Z_t \\ &\ge \mathrm{E}\int_0^{T\wedge\tau} \mathrm{e}^{-rt}\,\mathrm{d}Z_t.\end{aligned}$$

Letting $T \to \infty$,

$$\bar{V}(x) \ge \int_0^{\tau} \mathrm{e}^{-rt}\,\mathrm{d}Z_t.$$

Taking the supremum over all Z_t in the last inequality, we have

$$\bar{V}(x) \ge V^*(x).$$

Now we need to show that $\bar{V}(x) \le V^*(x)$. Taking the above control policy $U^*(x)$, let Z^* be the regulatory process at $x = x^*$. Repeating the previous argument, we have for any $T > 0$,

$$\begin{aligned}\bar{V}(x) &= \mathrm{E}(\mathrm{e}^{-rT\wedge\tau}\bar{V}(X_{T\wedge\tau})) \\ &\quad - \mathrm{E}\int_0^{T\wedge\tau} \mathrm{e}^{-rt}(\tfrac{1}{2}\sigma^2(U^*)^2\bar{V}''(X_t) + (\mu U^* - \alpha(U^*)^2 - \delta)\bar{V}(X_t) - r\bar{V}(X_t))\,\mathrm{d}t \\ &\quad + \mathrm{E}\int_0^{T\wedge\tau} \mathrm{e}^{-rt}\bar{V}'(X_t)\,\mathrm{d}Z_t^*,\end{aligned}$$

that is,

$$\bar{V}(x) = \mathrm{E}(\mathrm{e}^{-rT\wedge\tau}\bar{V}(X_{T\wedge\tau})) + \mathrm{E}\int_0^{T\wedge\tau} \mathrm{e}^{-rt}\,\mathrm{d}Z_t^*.$$

Taking $T \to \infty$, noting that $\mathbf{1}_{\{\tau<\infty\}}X_\tau = 0$ and that V is of linear growth in the neighbourhood of ∞, we deduce that the first term in the right-hand side of the above equation converges to 0, while the second converges to $\int_0^\tau \mathrm{e}^{-rt}\,\mathrm{d}Z_t^*$; hence we have our claim that

$$\bar{V}(x) \le V^*(x).$$

Now our optimal control $(U^*(x), Z^*(x))$ is clear. Let $U^*(x)$ be as given by (13) and let $Z^*(x)$ be the so-called 'regulatory process' (cf. [9]) at point x^*: at any x bigger than x^*, take profit of $x - x^*$ immediately by applying a local time control—namely by pushing it as hard as possible back to x^*.

2.3. The case $\delta = 0$

This section is devoted to determining the value function when $\delta = 0$ and the corresponding optimal strategy (U_t^*, Z_t^*). Compared to the case of $\delta > 0$, the solution to $\delta = 0$ is more explicit. Moreover, the value function, as a function of δ, changes drastically as δ approaches 0. When $\delta = 0$, the value function $V^*(x)$ is singular at $x = 0$, which introduces certain technical difficulties in proving the verification theorem.

When $\delta = 0$, (2) can be rewritten as:

$$\mathrm{d}X_t = U(t)(\mu - \alpha U(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W_t - \mathrm{d}Z_t.$$

And the corresponding HJB equation is

$$\max_{U,Z}[(1 - V_x), -rV(x) + (\mu U(x) - \alpha U^2(x))V_x + \tfrac{1}{2}\sigma^2 U(x)^2 V_{xx}] = 0.$$

Notice that $\mu \geq 0$ so that the capital X_t enjoys a nonnegative drift when $U(t) \leq \mu/\alpha$. However, the maximum of $y(\mu y - \alpha y)$ is $4\alpha/\mu^2$. Therefore, at $x = x^*$, $U(x) = \mu/(2\alpha)$, and the capital process X_t reaches its largest possible drift. Therefore, once $x > x^*$, we should push x as quickly as possible back to x^* in order to keep the maximum drift. Thus, we are looking for a smooth function $\bar{V}(x)$ such that

$$\bar{V}(x) = \begin{cases} V_1(x) & x \leq x^*, \mathrm{d}Z = 0, \\ V_2(x) & x > x^*, \mathrm{d}Z > 0, U(x) \equiv \mu/(2\alpha), \end{cases}$$

with $x^* = \inf\{x > 0 \mid U(x^*) = \mu/(2\alpha)\}$, for some control function $U(x)$.

Repeating similar calculations as in the previous section, we obtain the following relations for $V_1(x)$ and $U(x)$ for $x \leq x^*$:

$$\begin{aligned} -rV_1 + (\mu U(x) - \alpha U^2(x))V_{1x} + \tfrac{1}{2}\sigma^2 U^2(x) V_{1xx} &= 0, \\ (\mu - 2\alpha U(x))V_{1x} + \sigma^2 U(x) V_{1xx} &= 0. \end{aligned} \tag{16}$$

Solving (16), we obtain

$$\mu U(x) V_{1x} = 2rV_1. \tag{17}$$

Now, $V(0) = 0$ leads to $U(0) = 0$ such that

$$\begin{aligned} 2r - \mu U'(x) &= -\frac{\mu}{\sigma^2}(\mu - 2\alpha U(x)), \\ U|_{x=0} &= 0. \end{aligned}$$

Solving for $U(x)$, we get

$$U(x) = \frac{\mu^2 + 2r\sigma^2}{2\alpha\mu}\left(1 - \exp\left(-\frac{2\alpha}{\sigma^2}x\right)\right), \qquad x < x^*. \tag{18}$$

Now, from

$$x^* = \inf\{x > 0 \mid U(x) = 2\alpha/\mu\},$$

we obtain

$$x^* = \frac{\sigma^2}{2\alpha} \ln\left(1 + \frac{\sigma^2}{2r\sigma^2}\right).$$

Solving (17) by replacing $U(x)$ in (17) with (18), we have for $V_1(x)$,

$$\begin{aligned} V_1(x) &= \exp\left(d + \frac{2r\sigma^2}{\mu^2 + 2r\sigma^2} \ln\left(\exp\left(\frac{2\alpha}{\sigma^2}x\right) - 1\right)\right) \\ &= \mathrm{e}^d \left(\exp\left(\frac{2\alpha}{\sigma^2}x\right) - 1\right)^{2r\sigma^2/(\mu^2+2r\sigma^2)}, \end{aligned}$$

where d is a constant defined by the boundary conditions of $V - 1(x)$. Now, the 'smooth fit' at $x = x^*$ dictates that $V_{1x}(x^*) = 1$, which leads to

$$\mathrm{e}^d = \frac{\mu^2}{4\alpha r} \left(\frac{2r\sigma^2}{\mu^2}\right)^{2r\sigma^2/(\mu^2+2r\sigma^2)}.$$

Consequently, the next result follows.

Proposition 4. *We have*

$$\lim_{x\to 0} \frac{\mathrm{d}V_1(x)}{\mathrm{d}x} = \infty.$$

In other words, when $\delta = 0$, the value function is singular at $x = 0$. In short, we have obtained the explicit optimal value function $V^*(x)$.

Theorem 3. *When $\delta = 0$, $\mu > 0$, $V^*(x) = \overline{V(x)}$, with*

$$\overline{V(x)} = \begin{cases} \dfrac{\mu^2}{4\alpha r}\left(\dfrac{2r\sigma^2}{\mu^2}\left(\exp\left(\dfrac{2\alpha}{\sigma^2 x} - 1\right)\right)\right)^{2r\sigma^2/(\mu^2+2r\sigma^2)}, & x < x^*, \\ x + \dfrac{\mu^2}{4\alpha r} - \dfrac{\sigma^2}{2\alpha} \ln\left(1 + \dfrac{\mu^2}{2r\sigma^2}\right), & x > x^*, \end{cases}$$

where

$$x^* = \frac{\sigma^2}{2\alpha} \ln\left(1 + \frac{\sigma^2}{2r\sigma^2}\right),$$

Although the proof of Theorem 3 is essentially in the same spirit as that of Theorem 2, the fact that when $\delta = 0$, the value function is singular at 0 (from Proposition 4) requires the proof to be modified.

Let τ_ε be the first time where $X(t) = \varepsilon$. Then from the generalized Itô formula (cf. [9]), we have

$$\begin{aligned} \bar{V}(x) &= \mathrm{E}(\mathrm{e}^{-rT\wedge\tau_\varepsilon} \bar{V}(X_{T\wedge\tau_\varepsilon})) \\ &\quad - \mathrm{E}\int_0^{T\wedge\tau_\varepsilon} \mathrm{e}^{-rt}(\tfrac{1}{2}\sigma^2 U^2 \bar{V}''(X_t) + (\mu U - \alpha U^2 - \delta)\bar{V}'(X_t) - rV(X_t))\,\mathrm{d}t \\ &\quad + \mathrm{E}\int_0^{T\wedge\tau_\varepsilon} \mathrm{e}^{-rt}\bar{V}'(X_t)\,\mathrm{d}Z_t^{\mathrm{c}} - \mathrm{E}\sum_{0\le t\le T\wedge\tau_\varepsilon} \mathrm{e}^{-rt}(\bar{V}(X_t) - \bar{V}(X_{t-})). \end{aligned}$$

Since

$$\bar V(X_t) - \bar V(X_{t-}) = \bar V(X_{t-} - (Z_t - Z_{t-})) - \bar V(X_{t-}) \geq -(Z_t - Z_{t-}),$$

we get

$$\begin{aligned}\bar V(x) &= \mathrm{E}(\mathrm{e}^{-rT\wedge\tau_\varepsilon}\bar V(X_{T\wedge\tau_\varepsilon})) + \mathrm{E}\int_0^{T\wedge\tau_\varepsilon} \mathrm{e}^{-rt}\,\mathrm{d}Z_t^{\mathrm{c}} + \mathrm{E}\sum_{0\leq t\leq T\wedge\tau_\varepsilon} \mathrm{e}^{-rt}(Z_t - Z_{t-})\\ &\geq \mathrm{E}(\mathrm{e}^{-rT\wedge\tau_\varepsilon}\bar V(X_{T\wedge\tau_\varepsilon})) + \mathrm{E}\int_0^{T\wedge\tau_\varepsilon} \mathrm{e}^{-rt}\,\mathrm{d}Z_t\\ &\geq \mathrm{E}\int_0^{T\wedge\tau_\varepsilon} \mathrm{e}^{-rt}\,\mathrm{d}Z_t.\end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$\bar V(x) \geq \mathrm{E}\int_0^{T\wedge\tau} \mathrm{e}^{-rt}\,\mathrm{d}Z_t.$$

Now, letting $T \to \infty$ yields

$$\bar V(x) \geq \int_0^{\tau} \mathrm{e}^{-rt}\,\mathrm{d}Z_t.$$

Taking the supremum over all Z_t in the last inequality, we have

$$\bar V(x) \geq V^*(x).$$

On the other hand, taking the above control policy $U^*(x)$, let Z^* be the local time at $x = x^*$. Then repeating the previous argument, we have, for any $T > 0$,

$$\begin{aligned}\bar V(x) = {} & \mathrm{E}(\mathrm{e}^{-rT\wedge\tau_\varepsilon}\bar V(X_{T\wedge\tau_\varepsilon}))\\ & - \mathrm{E}\int_0^{T\wedge\tau_\varepsilon} \mathrm{e}^{-rt}(\tfrac{1}{2}\sigma^2(U^*)^2\bar V''(X_t) + (\mu U^* - \alpha(U^*(X_t))^2)\bar V(X_t) - r\bar V(X_t))\,\mathrm{d}t\\ & + \mathrm{E}\int_0^{T\wedge\tau_\varepsilon} \mathrm{e}^{-rt}\bar V'(X_t)\,\mathrm{d}Z_t^*.\end{aligned}$$

Therefore,

$$\begin{aligned}\bar V(x) &= \mathrm{E}(\mathrm{e}^{-rT\wedge\tau_\varepsilon}\bar V(X_{T\wedge\tau_\varepsilon})) + \mathrm{E}\int_0^{T\wedge\tau_\varepsilon} \mathrm{e}^{-rt}\,\mathrm{d}Z_t^*\\ &\leq \bar V(\varepsilon) + \mathrm{E}(\mathrm{e}^{-rT\wedge\tau}\bar V(X_{T\wedge\tau})) + \mathrm{E}\int_0^{T\wedge\tau} \mathrm{e}^{-rt}\,\mathrm{d}Z_t^*.\end{aligned}$$

Taking $\varepsilon \to 0$, we get

$$\bar V(x) \leq \mathrm{E}(\mathrm{e}^{-rT\wedge\tau}\bar V(X_{T\wedge\tau})) + \mathrm{E}\int_0^{T\wedge\tau} \mathrm{e}^{-rt}\,\mathrm{d}Z_t^*,$$

since $\bar V(x)$ is a continuous function.

Now taking $T \to \infty$ and noting that $\mathbf{1}_{\{\tau<\infty\}}X_\tau = 0$ and that V is of linear growth in the neighbourhood of ∞, i.e.

$$\mathrm{E}\bar V(X_{t\wedge\tau}) \leq x + K_1 t,$$

for some constant K_1, we deduce that the first term in the right-hand side of the above equation converges to 0, while the second converges to $\int_0^\tau \mathrm{e}^{-rt}\,\mathrm{d}Z_t^*$. Thus,

$$\bar{V}(x) \le \mathrm{E}\int_0^{T\wedge\tau} \mathrm{e}^{-rt}\,\mathrm{d}Z_t^* \le V^*(x).$$

This shows that $\bar{V}(x)$ is in fact achievable.

2.4. Ruin probability

To calculate the ruin probability is to study the behaviour of a Markov process near the boundary of 0. Here, we restrict ourselves to consider $\mathrm{P}\{\tau \mid X(0) = x\}$ for any $x < x^*$, given the optimal control policy $(U^*(x), Z^*(x))$ in the previous section such that, for $X_t < x^*$,

$$\mathrm{d}X_t = (\mu U^*(X_t) - \alpha(U^*(X_t))^2 - \delta)\,\mathrm{d}t + \sigma U^*(X_t)\,\mathrm{d}W_t.$$

2.4.1. *Ruin probability for* $\delta > 0$. As a consequence of Theorem 2, we see that under the optimal strategy $U^*(x)$, with probability 1, the company goes bankrupt in finite time. This is because $U^*(x) > 0$ and it is bounded below by $2\delta/\mu$, and $\mu U^*(x) - \alpha U^*(x)^2 - \delta$ is positive and has a finite upper bound. Thus, by the well-known fact that a Markov process with strictly positive diffusion term and bounded drift and with reflected barrier hits any point in a compact set in finite time with probability 1, we have the next corollary.

Corollary 1. *When* $\delta \neq 0$*, under the optimal policy* $U^*(x)$*,* $\mathrm{P}\{\tau < \infty\} = 1$.

2.4.2. *Ruin probability for* $\delta = 0$. Adopting the terminology in Section 6, Chapter 15 of [11], we start by checking $S(x)$ for $x < x^*$, where

$$S(x) = \int_0^x s(y)\,\mathrm{d}y = \int_0^x \exp\biggl(-2\int^y \frac{\mu U^*(\theta) - \alpha(U^*(\theta))^2}{(\sigma U^*(\theta))^2}\,\mathrm{d}\theta\biggr)\,\mathrm{d}y.$$

If $S(x) = \infty$ for some $x < x^*$, then the ruin probability is 0. Otherwise, it means that 0 is an attracting point, and we need to further check another index function $\Sigma(0)$. If $\Sigma(0) = \infty$, then again the ruin probability is 0 although 0 is an attracting point.

Incorporating the expression for $U^*(x)$ in (18), we see that $s(y) = \exp(-2H(y))$, where

$$\begin{aligned}
H(y) &= \int^y \frac{\mu U^*(y) - \alpha(U^*(y))^2}{(\sigma U^*(y))^2}\,\mathrm{d}y \\
&= \int^y \frac{\mu}{\sigma^2 U^*(y)}\,\mathrm{d}y - \int^y \frac{\alpha}{\sigma^2}\,\mathrm{d}y \\
&= \frac{-\alpha y}{\sigma^2} + \int^y \frac{2\alpha\mu^2}{\sigma^2(\mu^2 + 2r\sigma^2)}\frac{1}{1 - \exp(-2\alpha y/\sigma^2)}\,\mathrm{d}y \\
&= \frac{-\alpha y}{\sigma^2} + \frac{\mu^2}{\mu^2 + 2r\sigma^2}\ln\biggl(\exp\biggl(\frac{2\alpha y}{\sigma^2}\biggr) - 1\biggr) + C.
\end{aligned}$$

Hence

$$s(y) = \mathrm{e}^{-2C}\frac{\exp(2\alpha y/\sigma^2)}{(\exp(2\alpha y/\sigma^2) - 1)^{2\mu^2/(\mu^2 + 2r\sigma^2)}}.$$

Now, when $\mu^2 \neq 2r\sigma^2$, we see that

$$S(x) = \mathrm{e}^{-2C} \int_0^x \frac{\exp(2\alpha y/\sigma^2)}{(\exp(2\alpha y/\sigma^2) - 1)^{2\mu^2/(\mu^2+2r\sigma^2)}}\,\mathrm{d}y$$
$$= \mathrm{e}^{-2C} \frac{\mu^2 + 2r\sigma^2}{2r\sigma^2 - \mu^2} \frac{\sigma^2}{2\alpha} \left(\exp\left(\frac{2\alpha}{\sigma^2} x \right) - 1 \right)^{(2r\sigma^2-\mu^2)/(\mu^2+2r\sigma^2)} \Big|_0^x .$$

If $\mu^2 > 2r\sigma^2$, then $\exp(2\alpha l/\sigma^2) - 1 \to 0$ as $l \to 0$, thus

$$\left(\exp\left(\frac{2\alpha l}{\sigma^2} \right) - 1 \right)^{(2r\sigma^2-\mu^2)/(\mu^2+2r\sigma^2)} \to \infty,$$

which means that $S(x) = \infty$. Similarly, when $\mu^2 = 2r\sigma^2$,

$$S(x) = \mathrm{e}^{-2C} \frac{\sigma^2}{2\alpha} \ln\left(\exp\left(\frac{2\alpha x}{\sigma^2} \right) - 1 \right) \Big|_0^x$$

and

$$\ln\left(\exp\left(\frac{2\alpha l}{\sigma^2} \right) - 1 \right) \to -\infty \quad \text{as } l \to 0;$$

therefore, $S(x) = \infty$.

It is easy to see that when $\mu^2 < 2r\sigma^2$, $S(x) < \infty$. However, when $\mu^2 < 2r\sigma^2$, checking that $\Sigma(0^+) = \lim_{l\to 0} \Sigma(l)$ yields

$$\lim_{l\to 0} \Sigma(l) = \lim_{l\to 0} \int_l^x \left[\int_l^\theta s(\eta)\,\mathrm{d}\eta \right] \frac{1}{(\sigma U^*(\theta))^2 s(\theta)}\,\mathrm{d}\theta$$
$$= \lim_{l\to 0} C \int_l^x \left[\int_l^\theta \frac{\exp(2\alpha\eta/\sigma^2)}{(\exp(2\alpha\eta/\sigma^2) - 1)^{2\mu^2/(\mu^2+2r\sigma^2)}} \right] \mathrm{d}\eta$$
$$\times \frac{\exp(2\alpha\theta/\sigma^2)}{(\exp(2\alpha\theta/\sigma^2) - 1)^{2-2\mu^2/(\mu^2+2r\sigma^2)}}\,\mathrm{d}\theta$$
$$= A - \lim_{l\to 0} \ln\left(\exp\left(\frac{2\alpha l}{\sigma^2} \right) - 1 \right) = \infty.$$

Here A is bounded.

Hence, by Lemma 6.2 of [11, p. 230], we have the following corollary.

Corollary 2. *When* $\delta = 0$, $\mathrm{P}\{\tau < \infty\} = 0$ *under the optimal strategy* $U^*(x)$. *In particular,* 0 *is an attractive and unattainable point when* $\mu^2 < 2r\sigma^2$.

It is worth pointing out that, in [12] and [14], the statement that a company goes bankrupt in finite time with probability 1 (in their linear models) is not very precise. If we start from the left-most region in [14], there is a positive ruin probability, while if we start elsewhere, the ruin probability is 0. The reason is that when X_t is in the neighbourhood of 0, $c_1 U(x) < x < c_2 U(x)$, which makes the diffusion process a geometric Brownian motion, which never hits 0. This can also be derived via calculating functions $S(x)$ and $\Sigma(0)$ as above.

3. Conclusions

In this paper, we investigated a profit/dividend optimization problem for a company with internal competition and constant debt. Our result for the special case of $\delta = 0$ shows that neither the linearity assumption in [12] and [14] nor the monotonicity assumption in [3] is necessary. We also deal with the case $\delta > 0$ which was not considered previously.

Our explicit solutions enable us to gain a fair insight into the structure of the solution and the associated control policies. For example, our solution shows interesting discontinuities for the parameter δ, in terms of the ruin probability and of the 'singularity' features at 0 for the value functions.

Our techniques can be extended to a class of return functions broader than the quadratic form we considered in this paper, and can be applied to risk-management problems in re-insurance business in which insurance companies pay part of the premium to a re-insurance company in order to divert part of the business risk. The commonly used diffusion model is the continuous counter-part of the classical Cramér–Lundberg model, with an added parameter of insurance premium. Here the risk control U_t is translated into the re-insurance percentage, and hence U_t is bounded such that $0 \leq a \leq U_t \leq b \leq 1$ [8].

It would also be interesting to study the problem by adding a firing process and the associated cost functions for firing and hiring as in [14] for the linear case. Although we expect that the analysis would be similar, we speculate that the solution structure would be quite complicated.

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