Modeling the Recovery Rate in a Reduced Form Model*

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Abstract

This paper provides a model for the recovery rate process in a reduced form model. After default, a firm continues to operate, and the recovery rate is determined by the value of the firm's assets relative to its liabilities. The debt recovers a different magnitude depending upon whether or not the firm enters insolvency and bankruptcy. Although this recovery rate process is similar to that used in a structural model, the reduced form approach is maintained by utilizing information reduction in the sense of Guo, Jarrow and Zeng (2005). Our model is able to provide analytic expressions for a firm's default intensity, bankruptcy intensity, and zero-coupon bond prices both before and after default.

KEY WORDS: credit risk, recovery rates, reduced form model, filtration reduction

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1 Introduction

The credit risk literature studies the valuation and hedging of defaultable financial securities (see Bielecki and Rutkowski (2002), Duffie and Singleton (2003) and Lando (2004) for reviews). In its valuation methodology, two quantities are crucial. The first is the *likelihood of default* (or the *default intensity*, if it exists), and the second is the *recovery rate* in the event of default. The majority of the credit risk research effort involving reduced form models has focused on modelling the stochastic process for the default intensity. Much less work has been done on modelling the recovery rate process itself (one exception is Bakshi, Madan and Zhang (2001)).[1] In these reduced form models, it is usually assumed that the recovery rate is either a constant proportion of the firm's debt value at the instant before default (called the "recovery of market value") or a constant proportion of an otherwise equivalent Treasury's value at default (called the "recovery of face value"), see Bielecki and Rutkowski (2002).

As evidenced by this discussion, the existing reduced form models price risky debt *prior to* default. The pricing of defaulted debt is outside the model's formulation. Yet, markets for defaulted debt exist.[2] Furthermore, these markets are significant in pricing credit derivatives because defaulted debt prices, at default, are the basis for computing recovery rates (see Moody's (2005), Altman, Brady, Resti, Sironi (2003), and Acharya, Bharath, Srinivasan (2004)).[3] With the recent expansion of the credit derivative markets,[4] understanding both defaulted debt prices and the realized recovery rate has become an important issue, especially given the possible introduction of recovery rate swaps (see *Credit Magazine*, "The Next Generation," June 12, 2005).

In light of this gap in the literature, the purpose of the paper is to present a model of the firm's defaulted debt prices and the realized recovery rate in the context of a reduced form model. Taking its insights from the structural

[1] Some structural models also explicitly model the recovery rate process, e.g. Merton (1974).

[2] See Altman, Brady, Resti, Sironi (2003) for a brief discussion of the distressed and defaulted debt markets.

[3] The alternative approach is to measure realized recovery rates, equal to the prices of the debt, but at the emergence of default/bankruptcy, see Moody's (1999) and Acharya, Bharath, Srinivasan (2004).

[4] See Creditflux, issue 38, October 1, 2004 entitled " Surveys confirm rapid market growth."

approach to credit risk, we model the realized recovery rate using the firm's assets and liabilities. But, in order to retain the reduced form structure, we also employ the information reduction methodology of Guo, Jarrow and Zeng (2005). Combined, this generates an extended reduced form model, where the recovery rate process is modelled explicitly in terms of the firm's assets and liabilities. Our approach is consistent with a recent paper by Jarrow and Purnanandam (2004) who study risk management in a model where default, insolvency and bankruptcy are distinct economic conditions of the firm.

Our model can be used to quantify the firm's default process, recovery rate process, and risky debt price, both prior to and after default. These processes are quantified under two information structures related to the firm's asset value: that held by the firm's management (complete information), and that held by the market (partial information). Partial information is characterized by delayed knowledge of the firm's asset value. In our model, default is necessary to trigger the recovery rate process. The recovery rate process, if triggered, depends on the firm's asset value. When in default, first, the debt's maturity changes to a random time representing the resolution time for the financial distress. Next, if the firm becomes insolvent (defined as the firm's asset value falling below an insolvency barrier) before the financial distress is resolved, then the firm enters bankruptcy. In bankruptcy, the debt is paid off at some fractional level per dollar owed. In contrast, if the firm remains solvent until the firm's financial distress is resolved (so that it is not in bankruptcy), then the debt is paid off at a higher fractional level, perhaps unity.

Crucial in understanding the default and recovery rate processes are the default and bankruptcy intensity processes. These characterize the likelihoods of entering default, and within default, of the firm becoming insolvent and entering bankruptcy. Our model shows that when investors have complete information, and if the firm's asset value is below an insolvency threshold at the time of default, then the default and bankruptcy intensities are equal. However, if the asset value is above or equal to this critical level, then the default and bankruptcy intensities are distinct, and default does not necessarily lead to immediate bankruptcy. When investors have partial information, default and bankruptcy are both conceptually and analytically distinct. The formulas for pricing risky debt reflect these information differences and are quantified herein.

Our approach to modelling the recovery rate process generates a pricing formula for *defaulted* risky debt, in contrast to the traditional reduced form credit risk models, whose construction precludes this possibility. If the traditional model is calibrated to reflect the extended model's random recovery rate, then both the traditional and extended models will provide identical pre-default prices for the risky debt. Otherwise, their pre-default debt prices will differ.

An outline of the paper is as follows. Section 2 provides the general framework for analysis. Section 3 analyzes the bankruptcy time, while section 4 considers the recovery rate process and risky debt pricing. Section 5 concludes.

2 The General Framework

We consider a reduced form credit risk model for a firm's risky debt. Traded will be a term structure of default free zero-coupon bonds and a firm's risky zero-coupon bond. The firm's risky zero-coupon bond will represent a promised $1 to be paid at some future date T. If the firm defaults prior to time T, then there will be a recovery rate, between zero and one, paid per promised dollar. Traditional reduced form credit risk models assume that at default: (1) the model ends (default is an absorbing state) and (2) the recovery rate is known. The known recovery rate is either a constant proportion of the debt's value prior to default (called the "recovery of market value") or a constant proportion of an otherwise equivalent Treasury's value at default (called the "recovery of face value"), see Bielecki and Rutkowski (2002). Although convenient, these simplifying assumptions are not satisfied in practice. Firstly, default is not necessarily an absorbing state. Many firms enter default (and even bankruptcy) to emerge later as an operating entity. Secondly, at the time of default, a debt's recovery rate is not known, but represents a random variable, paid at some future (and random) date.

The purpose of this paper is to generalize the traditional reduced form credit risk models by including a stochastic recovery rate process in the event of default. Consistent with this objective, subsequent to default, there will be two economic states of the firm: solvency or insolvency. The firm can remain solvent and pay off the debt as promised[5], or the firm can reach insolvency,

[5]Although this payoff may occur after the debt's maturity date. If this is the case, then the relevant quantity at the debt's maturity is the present value of the debt's promised payment, from the date actually received to the debt's maturity date.

enter bankruptcy, and pay off only a fraction of the debt's value.[6] The notion that default, insolvency and bankruptcy are different states has precedence in the literature. This idea is explicit in Jarrow and Purnanandam (2004) and implicit in Robicheck and Myers (1966), Anderson and Sundaresan (1996), Mella-Barral and Perraudin (1997), Mella-Barral (1999), Fan and Sundaresan (2000) and Acharya, Huang, Subrahmanyam and Sundaram (2002).

To present this extension, we must first discuss the default model in its entirety; the task to which we now turn. Our analysis will be based on two representative continuous time stochastic processes for the firm's asset value: a regime switching model with continuous sample paths, and a diffusion model with jumps. We study these processes in order to obtain analytic solutions for the firm's default, bankruptcy and debt price processes. Analytic solutions facilitate intuition, comparative statics and empirical estimation. As discussed below, this structure is readily generalized, but at the expense of losing the analytic solutions.

2.1 The Firm's Asset Value Process

This section presents the two representative processes for the firm's asset value: a regime switching model and a jump diffusion process.

The Regime Switching Model Let $(X_t)_{t \geq 0}$ be the firm's asset value process that follows a diffusion process given by

$$dX_t = X_t \mu_{\epsilon(t)} dt + X_t \sigma_{\epsilon(t)} dW_t \tag{2.1}$$

where W is a standard one-dimensional Brownian motion, and $(\epsilon(t))_{t \geq 0}$ is a finite-state continuous-time Markov chain, independent of W, taking values $0, 1, \cdots, S-1$ with a known generator $(q_{ij})_{S \times S}$. Moreover, if $\gamma = \inf\{t > 0 : \epsilon(t) \neq \epsilon(0)\}$, then $P(\gamma > t | \epsilon(0) = i) = e^{-q_i t}$, where $q_i = \sum_{j \neq i} q_{ij}$. The drift and volatility coefficients $\mu(\cdot), \sigma(\cdot)$ are functions of ϵ. This dependency of the asset value's drift and volatility process on the state of the firm will be important, subsequently, for capturing the costs related to default (or the health of the firm).

Consistent with Jarrow, Lando and Turnbull (1997), one interpretation of the state space for this continuous time, time-homogenous Markov chain

[6] In bankruptcy, either liquidation (chapter 7) or reorganization (chapter 11) occurs, which results in a fractional payoff of the firm's debt obligations.

is that it represents the firm's credit ratings, with $S-1$ being the highest and 1 being the lowest. The last state 0, represents default. Under this interpretation, ϵ represents publicly available information.[7] Note that we do not necessarily require 0 to be an absorbing state. In general, the drift and volatility coefficients are more favorable in the non-default state than those in the default state. The simplest case is when $S=2$ where $\epsilon(t)=1,0$ correspond to "healthy" and "default," respectively.

In addition, for ease of exposition, ϵ is assumed to be time-homogeneous throughout. As will become clear, the more general case where ϵ is not time-homogeneous is a straightforward extension which adds little extra economic insight.

In the above construction, default is given by the random time

$$\tau = \inf\{t > 0 : \epsilon(t) = 0\}.$$

Consistent with the traditional reduced form credit risk models, the default time has an intensity, which is given by $\lambda_t = q_{\epsilon(t)0}$. This formulation of the default time is chosen for simplicity, in order to focus on the post default process of the firm. Instead, we could have modelled default as the first hitting time of the asset value to some barrier, a higher barrier than for insolvency (as defined below). This alternative approach is contained in Jarrow and Purnanandam (2004).

The Jump-diffusion Model In the jump-diffusion model, we let W and ϵ be defined as in regime switching model and denote by T_n the n-th jump time of ϵ. We further assign to each state i of ϵ $(0 \le i \le S-1)$ a positive random variable ξ_i representing the jump amplitude of the firm's asset value at state i. We assume that $(\xi_i)_{0\le i\le S-1}$, ϵ and W are all independent, with $P(\xi_i = 1) = 0$. The firm's asset value process X is assumed to satisfy

$$X_t = X_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \prod_{0<s\le t, \Delta\epsilon(s)\neq 0} \xi_{\epsilon(s)}. \tag{2.2}$$

Here $\Delta\epsilon(t) := \epsilon(t) - \epsilon(t-)$. Moreover, we assume that ξ_i has a known distribution function F_i, and $\Delta F_i(t) := F_i(t) - F_i(t-)$. Since 0 is the default state, we assume $P(\xi_0 \ge 1) = 0$, implying a downward pressure in the firm's asset value while default. Note that the asset value's drift and volatility (μ, σ) are

[7] The importance of this statement will become relevant in a subsequent section.

constants. This implies that the jump process formulation does not include the regime switching model as a special case.

As in the regime switching model, default is the random time

$$\tau = \inf\{t > 0 : \epsilon(t) = 0\},$$

which has an intensity $\lambda_t = q_{\epsilon(t)0}$.

2.2 Default, Insolvency, Bankruptcy, and Information

Unlike typical reduced form models, we emphasize the distinction between default, insolvency, and bankruptcy.

Default Default occurs when a firm misses or delays a promised payment on one of its financial liabilities or violates a liability's covenant, including the restructuring of the firm's liabilities. Default does not imply that the firm is insolvent nor enters bankruptcy[8], nor does it imply that all the firm's debt will not pay its promised payments. As mentioned earlier, the publicly available signal concerning the firm's health is represented by the state process $(\epsilon(t))_{t\geq 0}$, and the default time τ is

$$\tau = \inf\{t > 0 : \epsilon(t) = 0\}. \tag{2.3}$$

When default occurs, the firm's asset value process $(X_t)_{t\geq 0}$ changes. In default, the firm faces deadweight losses due to monitoring by the firm's liability holders, 3rd party costs, and possibly the institution of suboptimal investment decisions (see Bris, Welch and Zhu (2006) and references therein for estimates and a discussion of these various costs). These deadweight losses are reflected in changed drift and volatility coefficients $\mu(\cdot)$ and $\sigma(\cdot)$ in the regime switching model of expression (2.1) or a sudden downward jump in the firm's asset value in the jump diffusion model of expression (2.2).

Insolvency and Bankruptcy In default, there are two possible states of the firm, solvency and insolvency. Consistent with the structural approach

[8]Conversely, if a firm enters bankruptcy, by definition it will either liquidate (chapter 7) or reorganize (chapter 11), implying default. So, default and bankruptcy can happen simultaneously, however, they are different economic conditions of the firm (as discussed below).

to credit risk, once in default, insolvency occurs when the firm's asset value falls to a certain prescribed level x,[9] i.e.,

$$\{X_t < x\}.$$

This insolvency barrier x is inclusive of all deadweight costs incurred immediately at the onset of default.

In default, insolvency induces bankruptcy. Bankruptcy is a legal state of the firm where the debtor obtains court protection from the liability holders in order to resolve the firm's financial distress. In the U.S., the focus of our modeling, bankruptcy filings can be either under chapter 7 (liquidation) or chapter 11 (reorganization). In our model, insolvency provides the economic rationale for entering bankruptcy. Indeed, when the firm is insolvent, the firm's liabilities cannot be paid in full, and in order to resolve the liabilities holders claims with respect to the insufficient assets, court protection is needed. If the firm remains solvent, then bankruptcy is avoided and financial distress is handled out-of-court.

Formally, bankruptcy is defined to be the event

$$\{X_t < x, \epsilon(t) = 0\}.$$

The bankruptcy time is defined by

$$\tilde{\tau} = \inf\{t > 0 : X_t < x, \epsilon(t) = 0\}. \tag{2.4}$$

Note that if the firm is not in default, then the asset value hitting the barrier x does not induce bankruptcy. This is for two reasons: one, the barrier x reflects the additional deadweight costs paid at the onset of default (these are not incurred if not in default); and two, if not in default, then the creditors cannot force bankruptcy (liquidation and/or reorganization) when the barrier is breached. Furthermore, the firm's asset value process has more favorable drift and volatility coefficients in the non-default state, making eventual insolvency less likely.

[9] One can think of the stopping time as being determined by the firm's assets $\overline{X}_t$ and a stochastic boundary L_t, i.e., $\tau = \inf\{t > 0 : \overline{X}_t \leq L_t\}$. To obtain the characterization in the text, we take (depending upon ease of use) $X(t) \equiv \frac{\overline{X}_t}{L_t}$ or $\equiv \overline{X}_t - L_t$ and we have $\tau = \inf\{t > 0 : \frac{\overline{X}_t}{L_t} \leq 1\}$ or $= \inf\{\overline{X}_t - L_t \leq 0\}$.

Information (filtration) Structures We study the bankruptcy time under two different filtrations: *complete* and *partial* information. The complete information case corresponds to the information held by management, called investor $\mathcal{A}$. Investor $\mathcal{A}$ sees ϵ, the realization of the state of the firm (including default), and has complete (i.e., continuous) access to the firm's asset value process X. Investor $\mathcal{A}$ could also be interpreted as an informed trader or a government regulator.

The case of partial information corresponds to the information set held by the market, called investor $\mathcal{B}$. This information set determines market prices. Investor $\mathcal{B}$ knows the state of the firm, ϵ, but has limited access to the firm's asset value process X. In particular, we assume that investor $\mathcal{B}$ can only observe the firm's asset value at a times $t_1, t_2, \ldots, t_k, \ldots$, when the firm provides its quarterly report, and at the random times $T_1, T_2, \ldots, T_n, \ldots$, when the state of the firm changes. These random times could be due to the firm issuing press releases or through articles in the financial press. In the jump-diffusion models, additionally observed are the jump sizes of asset value at these random times. The interpretation is that jumps in the firm's asset value process are associated with announcements describing the circumstances surrounding the asset value change. Other information formulations are possible and left to subsequent research.

Following Guo, Jarrow and Zeng (2005), it is easy to see that the corresponding filtration (information) structure up to time t for investor $\mathcal{A}$ in both the regime switching and the jump models is the augmented natural filtration of (ϵ, X) such that $\mathcal{F}_t^{\mathcal{A}} = \sigma\{X_s, \epsilon(s), 0 \leq s \leq t\} \vee \mathcal{N}$, where $\mathcal{N}$ is the collection of all negligible sets. In contrast, investor $\mathcal{B}$'s filtration $\mathcal{F}_t^{\mathcal{B}}$ is the augmented minimal filtration generated by ϵ and the point processes of $1_{\{\tilde{\tau} \leq t\}}$, $\sum_{n=0}^{\infty} X_{t_k} 1_{\{t_k \leq t\}}$, and $\sum_{n=0}^{\infty} X_{T_n} 1_{\{T_n \leq t\}}$ for the regime switching model of expression (2.1); and by ϵ and the point processes of $1_{\{\tilde{\tau} \leq t\}}$, $\sum_{n=0}^{\infty} X_{t_k} 1_{\{t_k \leq t\}}$, $\sum_{n=0}^{\infty} \xi_{\epsilon(T_n)} 1_{\{T_n \leq t\}}$ and $\sum_{n=0}^{\infty} X_{T_n} 1_{\{T_n \leq t\}}$ for the jump diffusion model of expression (2.2).

3 The Bankruptcy Time

The first step in understanding the firm's recovery rate process is to characterize the bankruptcy time under the different information sets held by management (investor $\mathcal{A}$) and the market (investor $\mathcal{B}$). The techniques used in this paper for characterization of the bankruptcy time are extensions of those developed in Guo, Jarrow and Zeng (2005).

3.1 The Regime Switching Model

To simplify the presentation and without loss of generality, we assume that the firm starts from a non-default state, i.e., $\epsilon(0) \neq 0$.

Management (Investor $\mathcal{A}$) Recall that our bankruptcy time $\tilde{\tau} = \inf\{t > 0 : X_t < x,\ \epsilon(t) = 0\ \}$. Under management $\mathcal{A}$'s information set, if default happens (i.e., $\epsilon(t) = 0$) and if the value of X_t is known, then insolvency (and hence bankruptcy) is dictated by the level of X_t. More precisely, because of the continuous sample path of X, in the next instantaneous time interval $(t, t + \Delta t)$, the bankruptcy intensity is nonzero if and only if $X_t < x$. If $X_t < x$, then default immediately leads to bankruptcy, and the bankruptcy intensity at time t equals the default intensity $q_{\epsilon(t)0}$. However, if $X_t \geq x$, then the default and bankruptcy intensities are different. In this case the bankruptcy intensity is less than the default intensity $q_{\epsilon(t)0}$ because there is a positive probability that bankruptcy can be avoided.

According to Meyer's previsibility theorem, (see Rogers and Williams (2000)), $\tilde{\tau} = \min(\tilde{\tau}_\Lambda, \tilde{\tau}_{\Lambda^c})$ has a totally inaccessible component $\tilde{\tau}_\Lambda$ and a totally accessible component $\tilde{\tau}_{\Lambda^c}$ (see Appendix A for more details on the decomposition of a general stopping time into its predictable and totally inaccessible components) where the stopping time $\tau_\Lambda \equiv \begin{cases} \tau \text{ if } \omega \in \Lambda \\ \infty \text{ if } \omega \in \Lambda^c \end{cases}$, see Rogers and Williams (2000, page 11). The next theorem characterizes this decomposition.

Theorem 3.1. Assume expression (2.1). The bankruptcy time $\tilde{\tau}$ for managment (investor $\mathcal{A}$) has a totally inaccessible component $\tilde{\tau}_\Lambda$, with intensity $d_t^{R,\mathcal{A}}$, given by

$$d_t^{R,\mathcal{A}} = 1_{\{\tilde{\tau}>t,\ \epsilon(t)\neq 0\}} q_{\epsilon(t)0} 1_{\{X_t \leq x\}}. \tag{3.1}$$

The above theorem has an intuitive interpretation. Viewing the bankruptcy time $\tilde{\tau}$ as the first hitting time of the set $(-\infty, x) \times \{0\}$ for the joint process $(X_t, \epsilon(t))$, then at any given time $t < \tilde{\tau}$, in order for $\tilde{\tau} \in (t, t + \Delta t)$, there are two possibilities: (1) $X_t \geq x, \epsilon_t = 0$, and (2) $X_t < x$ but with $\epsilon(t) = j \neq 0$. For case (1), with probability $1 - q_{00}\Delta t \sim 1$, $\epsilon(t)$ remains at 0 and in the meantime $X(\cdot) \downarrow x$ in a continuous fashion. In this case the first hitting time is realized in a predictable fashion. This predictable component does not contribute to the bankruptcy intensity of $\tilde{\tau}$ because it relates to the predictable component. Case (2) generates $\tilde{\tau}_\Lambda$. Here the bankruptcy arrival is due to a sudden jump to $\epsilon(s) = 0$ for some $s \in (t, t + \Delta t)$. This jump time is totally inaccessible, with a rate of q_{j0} equal to the default arrival rate.

The Market (Investor $\mathcal{B}$) In contrast to the case for management (investor $\mathcal{A}$) where the default time equals the bankruptcy time if $X_t < x$, the bankruptcy time is conceptually and analytically different from the default time under partial information as held by the market. First, the filtration $\mathcal{F}^{\mathcal{B}}$ is of a "delayed" type as in the general framework of Guo, Jarrow and Zeng (2005), and delayed information induces an intensity for bankruptcy. This intensity, denoted as $d_t^{R,\mathcal{B}}$, can be derived via calculating

$$d_t^{R,\mathcal{B}} = \lim_{h \downarrow 0} \frac{P(t + h \geq \tilde{\tau} > t | \mathcal{F}_t^{\mathcal{B}})}{h}$$

(see Appendix B for details).

To understand this intensity process, recall that the market (investor $\mathcal{B}$) is provided at (current) time t with delayed information about the firm's asset value $X_{t_k \vee T_n}$ where t is assumed to satisfy $t_k \leq t < t_{k+1}$, $T_n \leq t < T_{n+1}$. Now, suppose that default occurs at time t and the most updated information about the firm's asset value is from time $t_k \vee T_n$ with $X_{t_k \vee T_n} < x$. Then, when $t = t_k \vee T_n$, clearly default triggers bankruptcy. However, if $t_k \vee T_n < t$, then with this "delayed" nature of information about X, default at time t does not necessarily lead to immediate bankruptcy. Indeed, there is a positive probability that bankruptcy is avoided at time t because the firm's asset value may have moved from below x at time $t_k \vee T_n$ to above x at (current time) t. On the other hand, even if default has not yet happened at time t, the fluctuation of X and the randomness of ϵ can lead to default and afterwards bankruptcy, but with less probability.

To formalize this intuition, note that given the structure of the Markov chain ϵ, the probability of ϵ changing more than two states between t and $t + \Delta t$ is of order $(\Delta t)^2$. Therefore, in default, the probability of avoiding insolvency (and hence bankruptcy) can be easily computed via calculating the probability of the running minimum of X during $(t, t+\Delta t)$ staying above x given $\epsilon(t)$ and $X_{t_k \vee T_n}$. That is,

Theorem 3.2. Assume expression (2.1). The bankruptcy time $\tilde{\tau}$ for the market (investor $\mathcal{B}$) is totally inaccessible under filtration $\mathcal{F}_t^{\mathcal{B}}$. Moreover, if

$t \in [t_k, t_{k+1})$, then when $\tilde{\tau} > t$, the bankruptcy intensity is

$$d_t^{R,\mathcal{B}} = \begin{cases} -\frac{\psi_t(\theta_0, t-t_k\vee T_n, \frac{1}{\sigma_0}\log\frac{x}{X_{t_k\vee T_n}})}{\psi(\theta_0, t-t_k\vee T_n, \frac{1}{\sigma_0}\log\frac{x}{X_{t_k\vee T_n}})}, & \text{if } \epsilon(t) = 0, \\ q_{\epsilon(t)0}\Phi\left(\frac{\frac{1}{\sigma_{\epsilon(t)}}\ln\frac{x}{X_{t_k\vee T_n}} - \theta_{\epsilon(t)}(t-T_n\vee t_k)}{\sqrt{t-T_n\vee t_k}}\right), & \text{if } \epsilon(t) \neq 0, \end{cases}$$

where $\theta_i = \frac{\mu_i}{\sigma_i} - \frac{\sigma_i}{2}$, Φ is the distribution function of standard normal random variable, and

$$\psi(\theta, t, y) = P(\inf_{0\le s\le t} W_s^{(\theta)} > y) = 1 - \int_0^t \frac{|y|}{\sqrt{2\pi s^3}} e^{-\frac{(y-\theta s)^2}{2s}} ds \quad \text{for } y < 0,$$

with $W_t^{(\theta)} := W_t + \theta t$. ψ_t is the derivative of ψ w.r.t. the variable t.

The detailed derivations are in Appendix B.

It is important to point out that the bankruptcy intensity $d_t^{J,\mathcal{B}}$ depends on the typical independent variables used in empirical hazard rate estimation procedures for bankruptcy (see Chava and Jarrow (2004) and references therein). More precisely, as indicated in the explicit representation above, the bankruptcy intensity depends on the firm's health $\epsilon(t)$, the drift of the log(asset) price process $\mu_{\epsilon(t)} - \frac{\sigma_{\epsilon(t)}^2}{2}$, the volatility of the log(asset) price process $\sigma_{\epsilon(t)}$, and the firm's debt/asset value ratio $\frac{x}{X_t}$. As expected, as the state of the firm changes from healthy $\epsilon(t) \neq 0$ to default $\epsilon(t) = 0$, the bankruptcy intensity increases. As the drift of the asset price process $\mu_{\epsilon(t)} - \frac{\sigma_{\epsilon(t)}^2}{2}$ increases, the intensity decreases. As the volatility of the asset price process $\sigma_{\epsilon(t)}$ increases, the intensity increases. Finally, as the firm's debt/asset ratio increases, the firm's bankruptcy intensity also increases. All these comparative statics are as expected.

3.2 The Jump Diffusion Model

The methodology utilized in the previous section can be similarly applied to the jump diffusion model where the firm's asset value process X follows expression (2.2). We therefore summarize the main results here without repeating the proofs.

Management (Investor $\mathcal{A}$) Again, according to Meyer's previsibility theorem $\tilde{\tau} = \min(\tilde{\tau}_\Lambda, \tilde{\tau}_{\Lambda^c})$ has a totally inaccessible component $\tilde{\tau}_\Lambda$ and totally accessible component $\tilde{\tau}_{\Lambda^c}$.

Theorem 3.3. Assume expression (2.2). The totally inaccessible component $\tilde{\tau}_\Lambda$ has an intensity, denoted as $d_t^{J,\mathcal{A}}$. Let $\{\tilde{\tau} > t, \epsilon(t) \neq 0\}$, then

$$d_t^{J,\mathcal{A}} = q_{\epsilon(t)0} F_0(\frac{x}{X_t})$$

where F_0 is the distribution function of ξ_0.

The intuition for this theorem is similar to that for Theorem 3.1, except that here X_t is no longer a continuous process and therefore its jump component also contributes to the bankruptcy intensity. In fact, comparing the bankruptcy intensity $d_t^{J,\mathcal{A}}$ of the jump diffusion model with that of the regime switching model $d_t^{R,\mathcal{A}}$, we see that $d_t^{J,\mathcal{A}}$ consists of both the default arrival rate $q_{\epsilon(t)0}$ and the jump component $F_0(\frac{x}{X_t})$ from the diffusion process, while $d_t^{R,\mathcal{A}}$ only equals the default intensity $q_{\epsilon(t)0}$ because of the continuous sample path. In both cases, however, the nature of the bankruptcy "surprise" is a direct consequence of the discontinuity (or, the "exogenous" nature) of the underlying process, either from $\epsilon(t)$ or from the jump component of the asset process. (See Guo and Zeng (2007) for general discussions on Hunt processes).

The Market (Investor $\mathcal{B}$) Recall that in this case we are given a sequence of constant times $\{t_k\}_{k\geq 0}$. For any fixed time t, if $t_k \leq t < t_{k+1}$ and $T_n \leq t < T_{n+1}$, observed are X_{t_1}, $\cdots$, X_{t_k}, T_1, $\cdots$, T_n, $\xi_{\epsilon(T_1)}$, $\cdots$, $\xi_{\epsilon(T_n)}$, X_{T_1}, $\cdots$, and X_{T_n}. This is equivalent to observing W_{t_1}, $\cdots$, W_{t_k}, T_1, $\cdots$, T_n, $\xi_{\epsilon(T_1)}$, $\cdots$, $\xi_{\epsilon(T_n)}$, W_{T_1}, $\cdots$, and W_{T_n}.

Theorem 3.4. Assuming expression (2.2) and $\{\tilde{\tau} > t, t_k \leq t < t_{k+1}, T_n \leq t < T_{n+1}\}$, the bankruptcy intensity, denoted as $d_t^{J,\mathcal{B}}$, is given by

$$\begin{aligned}
d_t^{J,\mathcal{B}} &= -\frac{\psi_t(\theta, t - t_k \vee T_n, \frac{1}{\sigma}\log\frac{x}{X_{t_k \vee T_n}})}{\psi(\theta, t - t_k \vee T_n, \frac{1}{\sigma}\log\frac{x}{X_{t_k \vee T_n}})}, \quad \text{if } \epsilon(t) = 0, \\
d_t^{J,\mathcal{B}} &= q_{\epsilon(t)0} P(y\xi_0 e^{\sigma W_u^{(\theta)}} \leq x)|_{u = t - t_k \vee T_n, y = X_{t_k \vee T_n}} \\
&= q_{\epsilon(t)0} \int_0^1 F_0(dv) \Phi\left(\frac{\frac{1}{\sigma}\log\frac{x}{vX_{t_k \vee T_n}} - \theta(t - t_k \vee T_n)}{\sqrt{t - t_k \vee T_n}}\right), \quad \text{if } \epsilon(t) \neq 0.
\end{aligned}$$

Here $\theta = \frac{\mu}{\sigma} - \frac{\sigma}{2}$.

The above formula for the bankruptcy intensity illustrates how bankruptcy occurs. Given the delayed information of the firm's asset value at $t_k \vee T_n$, suppose that the firm is in a non-default state $\epsilon(t) \neq 0$. Here, in order to become insolvent, the firm's asset value X_t must drop below level x at time t from $X_{t_k \vee T_n}$ and default $\epsilon(t)$ must occur, with intensity $q_{\epsilon(t)0}$. The bankruptcy intensity, thus, comes from both the change in ϵ and the jump of the firm's asset value process: it is the arrival intensity of the default state times the probability that firm's asset value jumps across the insolvency barrier. In contrast, after default $\epsilon(t) = 0$, the bankruptcy intensity is only due to the unobservable movement of the firm's asset value process.

If the firm has not defaulted $\epsilon(t) \neq 0$ and if F_0 is continuous, then it is interesting to note that due to delayed information, the bankruptcy intensity for the market $\mathcal{B}$ is smaller than that for management $\mathcal{A}$: although both investors $\mathcal{A}$ and $\mathcal{B}$ know that the last observed asset value was below level x (at time $t_k \vee T_n$ and t respectively), for the market $\mathcal{B}$ the asset value has a positive probability of moving above x between time $t_k \vee T_n$ and t and has more chance of solvency. Of course, as $t \to t_k \vee T_n$, the bankruptcy intensity for the market $\mathcal{B}$ converges to that of the management $\mathcal{A}$.

4 The Recovery Rate Process and Risky Debt Pricing

We next consider the recovery rate process (after default) and the pricing of the firm's risky zero-coupon bonds for both management (investor $\mathcal{A}$) and the market (investor $\mathcal{B}$). With the introduction of the bankruptcy time and hence the differentiation between bankruptcy and default, it is natural to consider the probability of bankruptcy given default. This distinguishes our approach from the traditional risky debt pricing methodology.

For pricing purposes, we assume the existence of an equivalent martingale measure making the discounted risky zero-coupon bond's price a martingale. This is equivalent to assuming an arbitrage free market (see Duffie (1996) for details). In incomplete markets, defined by the regime-switching and jump diffusion models, it is well-known that such a martingale measure exists, but it is not unique (see Guo, Jarrow and Zeng (2005) for discussions on this issue). For the subsequent analysis, we fix a particular measure from this set of equivalent martingale measures, assuming that the market is in equilibrium. For simplicity of notation (and the exposition), we let the probability measure P underlying the regime switching model (expression (2.1)) and the jump-diffusion model (expression (2.2)) be this martingale measure. In addition, to simplify the presentation, we assume that the default free interest rate process is deterministic. A stochastic term structure of interest rate process could be introduced, however, it would significantly complicate many of the subsequent expressions.

4.1 Review: The Traditional Reduced Form Approach

For easy comparison, we shall briefly review the traditional reduced form approach to pricing risky debt. Consider a zero-coupon bond issued by the firm paying \$1 at time T if there is no default, and $\$\widetilde{R}$ at time T if the firm defaults prior to time T. For simplicity of exposition, we let $\widetilde{R} \in [0,1]$ be a constant, although it is possible to extend the analysis to an $\mathcal{F}_\tau^i$ measurable random variable $\widetilde{R}$. This formulation is called the *face value of debt recovery rate* process (see Jarrow and Turnbull (1995)). Similar results hold for other independent recovery rate processes (see Bielecki and Rutkowski (2002) for the relevant alternatives).

Let $\nu^i(t,T)$ denote the traditional value of the risky zero-coupon bond under information $i \in \{\mathcal{A}, \mathcal{B}\}$. Given this structure, the value of the firm's zero-coupon bond to either management (investor $\mathcal{A}$) or the market (investor $\mathcal{B}$) can be written as:

$$\nu^i(t,T) = e^{-\int_t^T r(s)ds} E[\widetilde{R} 1_{\{\tau \le T\}} + 1_{\{\tau > T\}} \mid \mathcal{F}_t^i]$$

$$= e^{-\int_t^T r(s)ds} [1 - (1 - \widetilde{R}) P(\tau \le T \mid \mathcal{F}_t^i)] \text{ for } t \le \tau$$

with $i \in \{\mathcal{A}, \mathcal{B}\}$. As indicated, the traditional approach prices the firm's risky debt prior to (or at) default. Note that the difference in prices between investor $\mathcal{A}$ and $\mathcal{B}$ is quantified by the difference between the conditional probabilities of default before time T for investors $\mathcal{A}$ and $\mathcal{B}$. The next proposition characterizes the relationship between these conditional probabilities.

In general Markov models, Guo, Jarrow and Zeng (2005) showed that there is a non-linear relation between the default probabilities under different filtration structures. (This is a generalization of the results by Collin-Dufresne, Goldstein and Helwege (2003) and by Jeanblanc and Valchev (2004)). More precisely,

Lemma 4.1: Let τ be any general stopping time. Let $(X_t)_{t\geq 0}$ be the underlying Markov process. Let $\mathcal{F}^{\mathcal{B}}_{\tau}$ be the corresponding minimal filtration expansions of the delayed type for which τ is a stopping time and $\mathcal{F}^{\mathcal{A}}$ is its natural filtrations. Then for any $s < t$,

$$P(\tau > T \mid \mathcal{F}^{\mathcal{B}}_t) = \frac{P(\tau > T|\mathcal{F}^{\mathcal{A}}_s)}{P(\tau > t|\mathcal{F}^{\mathcal{A}}_s)} 1_{\{\tau > t\}}. \tag{4.1}$$

Expression (4.1), substituted into $\nu^i(t,T)$ for $i \in \{\mathcal{A}, \mathcal{B}\}$, characterizes the difference between the firm's debt prices as viewed by management versus the market.

Recall that $P(\tau \leq T \mid \mathcal{F}^{\mathcal{B}}_0) = P(\tau \leq T \mid \mathcal{F}^{\mathcal{A}}_0)$, because at time 0 both management ($\mathcal{A}$) and the market ($\mathcal{B}$) have the same information. Moreover, when $s \uparrow t$, $P(\tau \leq T \mid \mathcal{F}^{\mathcal{B}}_s)$ and $P(\tau \leq T \mid \mathcal{F}^{\mathcal{A}}_s)$ converge to the same value. This relationship is independent of the risk-neutral measure under consideration, as long as the price process X remains Markovian under the pricing measure (This is true under quite general conditions, see Dynkin (1965) page 306 and Palmowski and Rolski (2002)). This shows that "less" information can be as good as complete information as long as it is updated. Finally, one does not necessarily have$P(\tau \leq T \mid \mathcal{F}^{\mathcal{A}}_t) > P(\tau \leq T \mid \mathcal{F}^{\mathcal{B}}_t)$ or $P(\tau \leq T \mid \mathcal{F}^{\mathcal{A}}_t) < P(\tau \leq T \mid \mathcal{F}^{\mathcal{B}}_t)$ because less information does not necessarily mean the conditional default probability is larger.[10]

4.2 The Recovery Rate Process

This section presents the recovery rate process for risky debt under our extension. This recovery rate process is based on the U.S. bankruptcy code and would have to be modified accordingly for bankruptcy in other countries. The recovery rate process is described in Figure 4.1. Considering the risky zero-coupon bond with maturity T, we assume that:

- the firm pays \$1 at time T if there is (and has been) no default,
- once default occurs ($\tau < T$), the maturity of the debt changes to a random variable $\overline{T}$ where $P(\overline{T} < \infty) = 1$. Here $\overline{T}$ represents the date where financial distress is resolved, either through liquidation (chapter

[10]It is worth noting that Proposition 4.1 and expression (4.1) apply to any general stopping time τ, including default times and bankruptcy times, under the change of filtrations.

7 bankruptcy) of the firm's assets or a reorganization (either in chapter 11 bankruptcy or out-of-court),[11]

- in default, the firm either becomes insolvent and bankruptcy occurs, i.e., $\tau \leq \tilde{\tau} \leq \overline{T}$; or stays solvent until $\overline{T}$, i.e., $\tilde{\tau} > \overline{T}$,
- if bankruptcy occurs before the financial distress is resolved $\tau \leq \tilde{\tau} \leq \overline{T}$, then the bond pays a realized recovery rate of \$$R$, and
- if the firm remains solvent up to the resolution of financial distress, then the bond pays a fractional recovery rate of \$$K$ where $R < K \leq 1$.

The reason why $R < K$ is due to the mechanics of financial distress. If the firm does not file for bankruptcy and restructures its liabilities out-of-court, then the firm avoids those costs associated with bankruptcy filings (see Diz and Whitman (2005)) and it avoids liquidation. Liquidation requires the filing of chapter 7 bankruptcy, and this is only executed when the firm's value as a going concern is less than if liquidated.[12] The appendix contains a refinement of the recovery rate process explicitly including liquidation (chapter 7 bankruptcy), reorganization (chapter 11 bankruptcy), and out of court restructuring to justify $R < K$.

Without loss of generality, we assume that K and R are paid at time T.[13] Again, for simplicity, we assume that both K and R are constants, although the analysis is easily extended for an $F^i_{\tilde{\tau}}$ measurable random variable K and an $F^i_{\overline{T}}$ measurable random variable R.

This recovery rate process is similar to that used in the structural approach to credit risk with a default barrier. The imposition of an exogenous recovery rate (a constant) if the barrier is breached accounts for typical violations of the creditor's absolute priority rules (see Jarrow and Turnbull (1995, page 58) for related discussions).

Estimates for realized recovery rate R after bankruptcy can be found in Moody's (1999) and Acharya, Bharath, Srinivasan (2004) where the realized

[11]The changed maturity of the debt is consistent with the observation that once a firm is in financial distress, the recovery rate is independent of the debt's maturity and coupon structure, and only depends on the debt's seniority, see Acharya, Bharath, Srinivasan (2004, footnote 10).

[12]This is due to the "best interest test" of the chapter 11 bankruptcy code, see White and Case (2000, p. 72).

[13]In actuality, K and R are paid at time $\overline{T}$. The transformation is then to replace K with $Ke^{-\int_T^{\overline{T}} r(s)ds}$ and R with $Re^{-\int_T^{\overline{T}} r(s)ds}$.

recovery rate is estimated as the price of the defaulted debt after emergence from bankruptcy.[14] Estimates of the time duration that firms spend in bankruptcy can be found in Moody's (1999).[15] Unfortunately, direct estimates for the recovery rate for out-of-court restructurings K are not readily available. However, given our model, implicit estimates of K can be obtained. This implicit estimation procedure will be discussed in a subsequent section.

If at time t, default has not happened, i.e., $\tau > t$, then we have that the time t value of the firm's zero-coupon bond to investor $\mathcal{A}$ or $\mathcal{B}$ is

$$\begin{aligned} V^i(t,T) &= e^{-\int_t^T r(s)ds}[E_t^i[1_{\{\tau>T\}}] + (K-R)E_t^i[1_{\{\tilde{\tau}>\overline{T},\tau\leq T\}}] \\ &\quad +RE_t^i[1_{\{\tau\leq T\}}]] \end{aligned}$$

where $E_t^i = E[\cdot|\mathcal{F}_t^i]$ is under a martingale measure with $i \in \{\mathcal{A},\mathcal{B}\}$. If defaulted by time t, but still solvent, then assuming the default state is absorbing,

$$V^i(t,T) = e^{-\int_t^T r(s)ds}[(K-R)P(\inf_{t\leq v\leq \overline{T}} X_v > x|\ \mathcal{F}_t^i) + R].$$

This later expression is useful for pricing *defaulted* debt in secondary market trading. The traditional approach is not formulated for this situation.

The key quantities to evaluate in the above expressions are the distributions of the default/bankruptcy times for regime switching and jump diffusion models. In some cases, closed-form analytical expressions are available.

4.2.1 Example: Jarrow-Lando-Turnbull

A special case of particular interest is when $\epsilon(t)$ follows the Markov chain as proposed by Jarrow, Lando and Turnbull (1997) with default being an absorbing state. Here, the state space of this continuous time, time-homogenous Markov chain represents the possible credit classes, with $S-1$ being the highest and 1 being the lowest. The last state 0, represents default. The generator $(q_{ij})_{S\times S}$ is of the form

[14] As mentioned in the introduction, empirical estimates of recovery rates are available in two forms: (1) as prices of the defaulted debt at the time of default, and (2) as realized recovery rates - prices of the defaulted debt at the emergence from bankruptcy.

[15] The median time in bankruptcy for bankrupt firms over the time period 1982-1997 is 1.15 years with a standard deviation of 1.2 years.

$$Q = (q_{ij})_{S\times S} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ q_{10} & -q_1 & q_{12} & \cdots & q_{1(S-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{(S-1)0} & q_{(S-1)1} & q_{(S-1)2} & \cdots & -q_{S-1} \end{pmatrix}$$

where $q_{ij} \geq 0$ for all i, j, and $q_i = \sum_{i\neq j} q_{ij}$. Since

$$P\{\epsilon(t+\Delta t) = j|\epsilon(t) = i\} = q_{ij}\Delta t + o(\Delta t), \text{ for } i \neq j,$$

the instantaneous arrival rate of default is given by $\lambda_t = q_{\epsilon(t)0}$. The simplest case is when $S = 2$ for which $\epsilon(t) = 1, 0$ correspond to "healthy" and "default," respectively.

In the following, we assume that the underlying price process follows the regime switching model of expression (2.1). The case of jump diffusion is essentially the same, except that the parameters θ_0 and σ_0 in the regime switching model will be replaced by θ and σ, respectively. For simplicity, we assume in this example $\overline{T} = T$.

For management $\mathcal{A}$, it is easy to compute the time t price $V^{\mathcal{A}}(t,T)$ for the risky bond with maturity T. Recall that x is the insolvency barrier, and consequently

Default by time t, i.e. $\tau \leq t$

$$V^{\mathcal{A}}(t,T) = \begin{cases} Re^{-\int_t^T r(s)ds} & X_t \leq x \\ e^{-\int_t^T r(s)ds}[(K-R)\psi(\theta_0, T-t, \frac{1}{\sigma_0}\log\frac{x}{X_t}) + R] & X_t > x. \end{cases}$$

No default prior to time t

$$\begin{aligned} V^{\mathcal{A}}(t,T) &= e^{-\int_t^T r(s)ds} E_t^{\mathcal{A}}[1\cdot 1_{\{\tau>T\}} + K\cdot 1_{\{\tilde{\tau}>T,\tau\leq T\}} + R\cdot 1_{\{\tilde{\tau}\leq T,\ \ \tau\leq T\}}] \\ &= e^{-\int_t^T r(s)ds}[E_t^{\mathcal{A}}[1_{\{\tau>T\}}] + RE_t^{\mathcal{A}}[1_{\{\tau\leq T\}}] \\ &\quad +(K-R)\int_t^T\int_x^\infty P(\tau\in ds, X_\tau\in dz|\mathcal{F}_t^{\mathcal{A}})\psi(\theta_0, T-s, \frac{1}{\sigma_0}\log\frac{x}{z})] \\ &= e^{-\int_t^T r(s)ds}[R + (1-R)P(\tau > T|\epsilon(t)\neq 0) \\ &\quad +(K-R)\int_t^T\int_x^\infty P(\tau\in ds, X_\tau\in dz|X_t, \epsilon(t)\neq 0)\psi(\theta_0, T-s, \frac{1}{\sigma_0}\log\frac{x}{z})]. \end{aligned}$$

In the above expression, $P(\tau > T|\epsilon(t) \neq 0)$ can be directly calculated following Jarrow, Lando, Turnbull (1997). $P(\tau \in ds, X_\tau \in dz|X_t, \epsilon(t) \neq 0)$ may be computed explicitly in some cases, for example, it can be calculated via inverting the analytical expression of the Laplace transform for both the regime switching model (see Guo (2001)) and the jump diffusion model with double exponential jumps (see Kou and Wang (2003)). When $S = 2$, the expression is further simplified to

$$\begin{aligned} & V^{\mathcal{A}}(t,T) \\ = \; & e^{-\int_t^T r(s)ds}[R + (1-R)e^{-q_{10}(T-t)} \\ & +(K-R)\int_t^T \int_x^\infty dzds \frac{q_{10}e^{-q_{10}(s-t)}}{z\sigma_1\sqrt{s-t}}\phi(\frac{\frac{1}{\sigma_1}\log\frac{z}{X_t} - \theta_1(s-t)}{\sqrt{s-t}}) \\ & \times\psi(\theta_0, T-s, \frac{1}{\sigma_0}\log\frac{x}{z})] \end{aligned}$$

where ϕ is the density function of a standard normal random variable.

For the market $\mathcal{B}$, define $u = t_k \vee T_n < t < t_{k+1} \wedge T_{n+1}$, we have,

Default by time t Assuming default is an absorbing state, here $\tau \leq u < t$, and $\epsilon_u = \epsilon_t = 0$,[16] so

$$V^{\mathcal{B}}(t,T) = \begin{cases} Re^{-\int_t^T r(s)ds} & X_u \leq x \\ e^{-\int_t^T r(s)ds}[(K-R)\psi(\theta_0, T-u, \frac{1}{\sigma_0}\log\frac{x}{X_u}) + R] & X_u > x. \end{cases}$$

No default prior to time t Here $u < t$, but $\epsilon_u = \epsilon_t$, and

$$\begin{aligned} V^{\mathcal{B}}(t,T) &= e^{-\int_t^T r(s)ds}E_t^{\mathcal{B}}[1\cdot 1_{\{\tau>T\}} + K\cdot 1_{\{\tilde{\tau}>T,\tau\leq T\}} + R\cdot 1_{\{\tilde{\tau}\leq T,\;\;\tau\leq T\}}] \\ &= e^{-\int_t^T r(s)ds}[E_t^{\mathcal{B}}[1_{\{\tau>T\}}] + RE_t^{\mathcal{B}}[1_{\{\tau\leq T\}}] \\ &\quad +(K-R)\int_t^T\int_x^\infty P(\tau\in ds, X_\tau\in dz|\mathcal{F}_t^{\mathcal{B}})\psi(\theta_0, T-s, \frac{1}{\sigma_0}\log\frac{x}{z})] \\ &= e^{-\int_t^T r(s)ds}[R + (1-R)P(\tau>T|\epsilon(t)\neq 0) \\ &\quad +(K-R)\int_t^T\int_x^\infty P(\tau\in ds, X_\tau\in dz|\mathcal{F}_t^{\mathcal{B}})\psi(\theta_0, T-s, \frac{1}{\sigma_0}\log\frac{x}{z})]. \end{aligned}$$

[16] Note that this assumes default is strictly before time t. On the default time $t = \tau$ we have that $V^{\mathcal{A}}(t,T) = V^{\mathcal{B}}(t,T)$. After the default time, however, these two values differ due to the delayed information for investor $\mathcal{B}$.

In particular,

$$
\begin{aligned}
& V^{\mathcal{B}}(t,T) \\
= \; & e^{-\int_t^T r(s)ds}[R + (1-R)e^{-q_{10}(T-t)} \\
& +(K-R)\int_t^T \int_x^\infty dz ds \frac{q_{10}e^{-q_{10}(s-t)}}{z\sigma_1\sqrt{s-t}}\phi(\frac{\frac{1}{\sigma_1}\log\frac{z}{X_u} - \theta_1(s-u)}{\sqrt{s-u}}) \\
& \times\psi(\theta_0, T-s, \frac{1}{\sigma_0}\log\frac{x}{z})]
\end{aligned}
$$

for $S = 2$.

Here, the prices for management $\mathcal{A}$ and the market $\mathcal{B}$ are different, and the latter is not a simple re-parametrization of the former, although the underlying calculation for both cases uses the Markovian structure of $(X_t, \epsilon(t))$. This is because in case $\mathcal{B}$ the information concerning X_t is delayed at $u < t$, and the no-default condition gives an updated observation of $\epsilon(\cdot)$. Therefore, the exponential time between jumps for the Markov chain $\epsilon(t)$ and the independence between $\epsilon(t)$ and W_t are essential for the derivation.

4.3 A Comparison

This section compares risky debt prices between our extended model and the traditional reduced form approach. In our setting, once the firm defaults, the "recovery" rate is determined by the firm's asset value process. In the traditional model, the "recovery" rate is given exogenously as a known constant at the default time. These observations characterize the conceptual difference between these two approaches.

4.3.1 At Default, i.e. $\tau = t$

At the default date, the two model prices are:

$$
\text{Traditional: } \nu^{\mathcal{B}}(\tau,T) = \nu^{\mathcal{A}}(\tau,T) = \widetilde{R}e^{-\int_\tau^T r(s)ds}
$$

and

$$
\begin{aligned}
& \text{Extended: } V^{\mathcal{B}}(\tau,T) = V^{\mathcal{A}}(\tau,T) \\
= & \begin{cases} Re^{-\int_\tau^T r(s)ds} & \text{if insolvent} \\ e^{-\int_\tau^T r(s)ds}[(K-R)P(\inf_{\tau\le u\le \overline{T}} X_u > x|\mathcal{F}_\tau^{\mathcal{B}}) + R] & \text{if solvent.} \end{cases}
\end{aligned}
$$

Note that at the default time τ, both management and the market (investors $\mathcal{A}$ and $\mathcal{B}$) have the same information. So, at default, for both the traditional model and the extended model, management and the market have the same prices.

Next, we compare these prices across the two different model types. In the extended model, the debt's price is a random variable at time τ if the firm is solvent, while in the traditional model it is a constant. Otherwise, the two model prices are both equal to the discounted value of the recovery rate. Hence, the two model prices will differ to the extent that the firm is solvent at the time of default.

4.3.2 Prior to Default, i.e. $\tau > t$

Prior to default, management and the market have different information sets, so their prices for risky debt will differ. For pricing traded debt, the relevant information set is the market's, hence, this section compares pre-default prices as viewed by investor $\mathcal{B}$. For market prices, the difference between the extended and traditional model prices depends on the calibration used for the traditional and the extended models.

For calibration purposes, readily available are the (average) market prices for defaulted debt at time of default τ, denoted M_τ, and at the time of emergence from bankruptcy, denoted M_∞. Estimates of the time duration that firms spend in financial distress $(\overline{T} - \tau)$ can be found in Moody's (1999) as well. A simple calibration procedure can be used to determine the parameters $\{R, K\}$ in the extended model. For the realized recovery rate if insolvency and bankruptcy occurs, the natural choice is

$$R = M_\infty.$$

And, K can be determined by solving for the parameter that equates the price of the debt at default (if solvent) to the model's price, i.e.

$$\begin{aligned} K &= M_\infty + \frac{M_\tau e^{\int_\tau^{\overline{T}} r(s)ds} - M_\infty}{P(\inf_{\tau \le u \le \overline{T}} X_u > x |\ \mathcal{F}_\tau^{\mathcal{B}})} \\ &= R + \frac{\widetilde{R} - R}{P(\inf_{\tau \le u \le \overline{T}} X_u > x |\ \mathcal{F}_\tau^{\mathcal{B}})}. \end{aligned}$$

If, consistent with the constant recovery rate assumption, the traditional model is calibrated to equate the defaulted debt's price to the model's price, i.e.

$$M_\tau = \widetilde{R}e^{-\int_\tau^T r(s)ds},$$

then the two model prices will differ prior to default. Indeed,

$$\begin{aligned} V^{\mathcal{B}}(t,T) - v^{\mathcal{B}}(t,T) &= e^{-\int_t^T r(s)ds}(R-\widetilde{R})P(\tau \le T|\mathcal{F}_t^{\mathcal{B}}) \\ &\quad + e^{-\int_t^T r(s)ds}(K-R)P(\tilde{\tau} > \overline{T}, \tau \le T|\mathcal{F}_t^{\mathcal{B}}) \end{aligned}$$

is non-zero because $\widetilde{R}$, R , and K differ.

It is possible, however, to calibrate the traditional model so that it equals the extended model's price prior to default. Letting $\widetilde{R}$ be an $\mathcal{F}_\tau^{\mathcal{B}}$-measurable random variable, we can set

$$\widetilde{R}e^{-\int_\tau^T r(s)ds} = V^{\mathcal{B}}(\tau,T)$$

$$= \begin{cases} Re^{-\int_\tau^T r(s)ds} & \text{if insolvent} \\ e^{-\int_\tau^T r(s)ds}[(K-R)P(\inf_{\tau \le u \le \overline{T}} X_u > x|\mathcal{F}_\tau^{\mathcal{B}}) + R] & \text{if solvent.} \end{cases}$$

Then, the traditional and extended model prices will equal prior to default, i.e. $v^{\mathcal{B}}(t,T) = V^{\mathcal{B}}(t,T)$.[17] This calibration necessarily relaxes the constant recovery rate assumption in the traditional model to an alternative random representation that is distinct from the constant recovery of market value assumption that is often employed in the existing literature.

5 Conclusion

In this paper, we propose a simple continuous time model for bankruptcy and the recovery rate process, useful for pricing risky debt both before and after default. These processes are quantified under two information structures related to the firm's asset value: that held by the firm's management (complete information), and that held by the market (partial information). Partial information is characterized by delayed knowledge of the firm's asset value. In our model, default is necessary to trigger the recovery rate process. The recovery rate process, if triggered, depends on the firm's asset value. If the debt matures before the firm becomes insolvent (defined as the firm's

[17]The proof of this fact is trivial. If default happens at time τ, the two debt prices are identical, and if default doesn't happen, they both pay \$1. Hence, their payoffs are identical with probability one.

asset value falling below an insolvency barrier), then the debt is paid in full or at some fractional level. The fractional recovery when the firm is solvent exceeds the amount that would be paid if the firm becomes insolvent and enters bankruptcy.

Our model shows that when investors have complete information, and if the firm's asset value is below an insolvency threshold at the time of default, then the default and bankruptcy intensity are equal. However, if the asset value is above or equal to this critical level, then the default and bankruptcy intensities are distinct, and default does not necessarily lead to immediate bankruptcy. When investors have partial information, default and bankruptcy are both conceptually and analytically distinct. This distinction leads to a more realistic model for pricing risky debt. In contrast to the traditional reduced form models, our model is also capable of pricing defaulted debt trading in the distressed debt market.

Appendix A: Accessible and Totally Inaccessible Parts of Stopping Times

In this section alone, with a bit abuse of notation, we will use τ for any general stopping time, including the default time and the bankruptcy time in the paper.

The most relevant result about the decomposition of a general stopping time τ is the following.

Theorem A.1. For every stopping time τ, there exists an $A \in \mathcal{F}_{\tau_-}$ such that $A \subset \{\tau < \infty\}$, and τ_A is accessible and τ_{A^c} is totally inaccessible. Such A is a.s. unique.

The detailed proof can be found from He, Wang and Yan (1992). However, a few remarks for this theorem are relevant here.

First, the key to the proof is to find the set A, from which the decomposition of τ into τ_A and τ_{A^c} is simple. Indeed, the two new stopping times τ_A and τ_{A^c} are the stopping time τ restricted on the set A and A^c, and are generally referred to as the accessible part and totally accessible part of τ, respectively.

Secondly, the proof of the existence of A is constructive, as follows. Define

$$\mathcal{H} = \{\cup_n \{S_n = \tau < \infty\} : (S_n)_{(n \geq 1)} \text{ is a sequence of predictable times}\}.$$

Clearly $\mathcal{H} \subset \mathcal{F}_{\tau_-}$, $\mathcal{H}$ is closed under the formation of countable unions. It is not hard to see that there exists an $A \in \mathcal{H}$ such that $A = esssup\mathcal{H}$, for which τ_A is accessible and τ_{A^c} is totally inaccessible.

Lastly, note that $\tau_A = \tau I_A + (+\infty) I_{A^c}$, $\tau \leq \tau_A$ and $\tau \leq \tau_{A^c}$. Here $A^c = \Lambda$ for Λ used in the main text.

Appendix B: Proof of Theorem 3.1

Proof: First, for ease of exposition, denote by $\mathcal{F}_{k,n}$ the σ-field generated by $W_{t_1}, \cdots, W_{t_k}, T_1, \cdots, T_n, \epsilon(T_1), \cdots, \epsilon(T_n)$ and $W_{T_1}, \cdots, W_{T_n}$; and denote $P(\cdot|\mathcal{F}(y_k, u_n, x_n, v_n))$ for $P(\cdot|W_{t_1} = y_1, \cdots, W_{t_k} = y_k, T_1 = u_1, \cdots, T_n = u_n, W_{T_1} = x_1, \cdots, W_{T_n} = x_n, \epsilon(T_1) = v_1, \cdots, \epsilon(T_n) = v_n)$.

Note that on the event $\{\tau > t, T_n \leq t < T_{n+1}\}$, when $T_1, \cdots, T_n$, $\epsilon(T_1), \cdots$, and $\epsilon(T_n)$ are known, there is a one-to-one correspondence between $(W_{t_1}, \cdots, W_{t_k}, W_{T_1}, \cdots, W_{T_n})$ and $(X_{t_1}, \cdots, X_{t_k}, X_{T_1}, \cdots, X_{T_n})$. We thus denote by x'_i the value of X_{T_i} $(1 \leq i \leq n)$ and by y'_j the value of X_{t_j} $(1 \leq j \leq k)$ given $\mathcal{F}(y_k, u_n, x_n, s_n)$. Finally, let $\theta_i = \frac{\mu_i}{\sigma_i} - \frac{\sigma_i}{2}$ $(0 \leq i \leq S-1)$, $X_t^{(i)} = \exp\{(\mu_i - \frac{\sigma_i^2}{2})t + \sigma_i W_t\}$ $(0 \leq i \leq S-1)$, and $(\mathcal{F}_t^W)_{t \geq 0}$ be the natural filtration of W.

First of all, by the Bayes' formula and the structure of the filtration $(\mathcal{F}_t)_{t \geq 0}$, we have, for $t \in [t_k, t_{k+1})$,

$$\begin{aligned} & P(t+h \geq \tilde{\tau} > t|\mathcal{F}_t^{\mathcal{B}}) \\ = \quad & 1_{\{\tilde{\tau} > t\}} \left(1 - \sum_{n \geq 0} 1_{\{T_{n+1} > t \geq T_n\}} \frac{P(\tilde{\tau} > t+h, T_{n+1} > t \geq T_n|\mathcal{F}_{k,n})}{P(\tilde{\tau} > t, T_{n+1} > t \geq T_n|\mathcal{F}_{k,n})} \right) \end{aligned}$$

Case (i): if $\epsilon(t) = 0$, then

$$
\begin{aligned}
& P(\tilde{\tau} > t, T_{n+1} > t \geq T_n | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
= \; & P(x_i' \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{T_i}^{(0)}} > x, i \leq n-1, v_i = 0, x_n' \inf_{u_n \leq s \leq t} \frac{X_s^{(0)}}{X_{u_n}^{(0)}} > x | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
& \times P(T_{n+1} > t | T_n = u_n, \epsilon(T_n) = v_n) \\
= \; & \begin{cases}
\text{if } u_n \geq t_k, \, P(x_i' \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1, v_i = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
\qquad e^{-q_0(t-u_n)} \psi(\theta_0, t - u_n, \frac{1}{\sigma_0} \log \frac{x}{x_n'}), \\
\text{if } u_n < t_k, \, P(x_i' \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1, v_i = 0, \\
\qquad x_n' \inf_{u_n \leq s \leq t_k} \frac{X_s^{(0)}}{X_{u_n}^{(0)}} > x | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
\qquad e^{-q_0(t-u_n)} \psi(\theta_0, t - t_k, \frac{1}{\sigma_0} \log \frac{x}{y_k'}).
\end{cases}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \frac{P(\tilde{\tau} > t+h, T_{n+1} > t \geq T_n | \mathcal{F}_{k,n})}{P(\tilde{\tau} > t, T_{n+1} > t \geq T_n | \mathcal{F}_{k,n})} \\
= \; & e^{-q_0 h} \frac{\psi(\theta_0, t + h - t_k \vee T_n, \frac{1}{\sigma_0} \log \frac{x}{X_{t_k \vee T_n}})}{\psi(\theta_0, t - t_k \vee T_n, \frac{1}{\sigma_0} \log \frac{x}{X_{t_k \vee T_n}})} + I + II,
\end{aligned}
$$

where

$$
\begin{aligned}
I &= \frac{P(\tilde{\tau} > t+h, T_{n+2} > t+h \geq T_{n+1} > t \geq T_n | \mathcal{F}_{k,n})}{P(\tilde{\tau} > t, T_{n+1} > t \geq T_n | \mathcal{F}_{k,n})}, \\
II &= \frac{P(\tilde{\tau} > t+h, t+h \geq T_{n+2} > T_{n+1} > t \geq T_n | \mathcal{F}_{k,n})}{P(\tilde{\tau} > t, T_{n+1} > t \geq T_n | \mathcal{F}_{k,n})}.
\end{aligned}
$$

To calculate I, it is clear that

$$
\begin{aligned}
& P(\tilde{\tau} > t+h, T_{n+2} > t+h \geq T_{n+1} > t \geq T_n | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
= & \sum_{j\neq 0} \int_t^{t+h} P(x'_i \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n, v_i = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
& \times P(T_{n+2} > t+h | T_{n+1} = u_{n+1}, \epsilon(T_{n+1}) = j) \\
& \times P(T_{n+1} \in du_{n+1}, \epsilon(T_{n+1}) = j | T_n = u_n, \epsilon(T_n) = 0) \\
= & \begin{cases} \text{if } u_n \geq t_k,\ P(x'_i \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1,\ v_i = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
\qquad \sum_{j \neq 0} \int_t^{t+h} e^{-q_j(t+h-u_{n+1})} q_0 e^{-q_0(u_{n+1}-u_n)} \frac{q_{0j}}{q_0} \psi(\theta_0, u_{n+1}-u_n, \frac{1}{\sigma_0} \log \frac{x}{x'_n}) du_{n+1}, \\
\text{if } u_n < t_k,\ P(x'_i \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1,\ v_i = 0, \\
\qquad x'_n \inf_{u_n \leq s \leq t_k} \frac{X_s^{(0)}}{X_{u_n}^{(0)}} > x | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
\qquad \sum_{j \neq 0} \int_t^{t+h} e^{-q_j(t+h-u_{n+1})} q_0 e^{-q_0(u_{n+1}-u_n)} \frac{q_{0j}}{q_0} \psi(\theta_0, u_{n+1}-t_k, \frac{1}{\sigma_0} \log \frac{x}{y'_k}) du_{n+1}. \end{cases}
\end{aligned}
$$

So I is

$$
\frac{\sum_{j\neq 0} \int_t^{t+h} e^{-q_j(t+h-u_{n+1})} q_{0j} e^{-q_0(u_{n+1}-T_n)} \psi(\theta_0, u_{n+1} - T_n \vee t_k, \frac{1}{\sigma_0} \log \frac{x}{X_{t_k \vee T_n}}) du_{n+1}}{e^{-q_0(t-T_n)} \psi(\theta_0, t - T_n \vee t_k, \frac{1}{\sigma_0} \log \frac{x}{X_{t_k \vee T_n}})},
$$

and $\frac{I}{h} \to \sum_{j\neq 0} q_{0j} = q_0$ as $h \downarrow 0$.

Similarly, $II \leq Ch^2$ for some constant C, and

$$
\lim_{h\downarrow 0} \frac{P(t+h \geq \tilde{\tau} > t | \mathcal{F}_t^{\mathcal{B}})}{h} = -\frac{\psi_t(\theta_0, t - t_k \vee T_n, \frac{1}{\sigma_0} \log \frac{x}{X_{t_k \vee T_n}})}{\psi(\theta_0, t - t_k \vee T_n, \frac{1}{\sigma_0} \log \frac{x}{X_{t_k \vee T_n}})}.
$$

Case (ii): if $\epsilon(t) \neq 0$, then

$$
\begin{aligned}
& P(\tilde{\tau} > t, T_{n+1} > t \geq T_n | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
= & P(x'_i \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1, v_i = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) e^{-q_{v_n}(t-u_n)},
\end{aligned}
$$

$$\begin{aligned}
& P(\tilde{\tau} > t+h, T_{n+1} > t+h > t \geq T_n | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
= \; & P(x_i' \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1, v_i = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) e^{-q_{v_n}(t+h-u_n)},
\end{aligned}$$

and

$$\begin{aligned}
& P(\tilde{\tau} > t+h, T_{n+2} > t+h \geq T_{n+1} > t \geq T_n | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
= \; & P(\tilde{\tau} > t+h, T_{n+2} > t+h \geq T_{n+1} > t \geq T_n, \epsilon(T_{n+1}) \neq 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
& + P(\tilde{\tau} > t+h, T_{n+2} > t+h \geq T_{n+1} > t \geq T_n, \epsilon(T_{n+1}) = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
= \; & I + II,
\end{aligned}$$

with

$$\begin{aligned}
I \; = \; & P(x_i' \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1, v_i = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
& \times \sum_{j \neq 0, \epsilon(T_n)} \int_t^{t+h} q_{\epsilon(T_n) j} e^{-q_{\epsilon(T_n)}(u-T_n)} e^{-q_j(t+h-u)} du,
\end{aligned}$$

and

$$\begin{aligned}
II \; = \; & \int_t^{t+h} P(x_i' \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1, v_i = 0, \epsilon(T_{n+1}) = 0, \\
& x_n' e^{(\mu_{v_n} - \frac{\sigma_{v_n}^2}{2})(u_{n+1}-u_n) + \sigma_{v_n}(W_{u_{n+1}} - W_{u_n})} \inf_{u_{n+1} \leq s \leq t+h} \frac{X_s^{(0)}}{X_{u_{n+1}}^{(0)}} > x | \mathcal{F}(y_k, u_n, x_n, v_n), T_{n+1} = u_{n+1} \\
& \times P(T_{n+2} > t+h | T_{n+1} = u_{n+1}) \times P(T_{n+1} \in du_{n+1} | T_n = u_n, \epsilon(T_n) = v_n) \\
= \; & \begin{cases}
\text{if } u_n \geq t_k, \; P(x_i' \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1, v_i = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
\qquad \frac{q_{v_n 0}}{q_{v_n}} \int_t^{t+h} e^{-q_0(t+h-u_{n+1})} q_{v_n} e^{-q_{v_n}(u_{n+1}-u_n)} \\
\qquad E[\psi(\theta_0, t+h-u_{n+1}, \frac{1}{\sigma_0} \log \frac{x}{x_n'} - \frac{\sigma_{v_n}}{\sigma_0} W^{(\theta_{v_n})}_{u_{n+1}-u_n})] du_{n+1} \\
\text{if } u_n < t_k, \; P(x_i' \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1, v_i = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
\qquad \int_t^{t+h} e^{-q_0(t+h-u_{n+1})} q_{v_n} e^{-q_{v_n}(u_{n+1}-u_n)} \frac{q_{v_n 0}}{q_{v_n}} \\
\qquad E[\psi(\theta_0, t+h-u_{n+1}, \frac{1}{\sigma_0} \log \frac{x}{y_k'} - \frac{\sigma_{v_n}}{\sigma_0} W^{(\theta_{v_n})}_{u_{n+1}-t_k})] du_{n+1}.
\end{cases}
\end{aligned}$$

Finally,

$$
\begin{aligned}
& P(\tilde{\tau} > t+h, t+h \geq T_{n+2} > T_{n+1} > t \geq T_n | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
\leq \; & P(x_i' \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1, v_i = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) e^{-q_{v_n}(t-u_n)} C h^2,
\end{aligned}
$$

for some constant C. Therefore,

$$
\begin{aligned}
& \lim_{h \downarrow 0} \frac{P(t+h \geq \tilde{\tau} > t | \mathcal{F}_t^{\mathcal{B}})}{h} \\
= \; & q_{\epsilon(t)0} \Phi \left(\frac{\frac{1}{\sigma_{\epsilon(t)}} \log \frac{x}{X_{T_n \vee t_k}} - \theta_{\epsilon(t)}(t - T_n \vee t_k)}{\sqrt{t - T_n \vee t_k}} \right).
\end{aligned}
$$

Appendix C: Justification for $R < K$

Bankruptcy filings can be under chapter 7 (liquidation) or chapter 11 (reorganization), see White and Case (2000). Let $\alpha \in [0,1]$ be the constant recovery rate under chapter 7 bankruptcy, where priority rules are adhered to. Let $\beta \in [0,1]$ be the constant recovery rate under chapter 11, where priority rules are often violated. Because of the "best interest test" provision in chapter 11 which mandates liquidation in bankruptcy, unless reorganization (the value of the firm as a going concern) is greater, we have that $\alpha < \beta$. Let γ be the constant recovery rate in an out-of-court restructuring of the firm's liabilities. Due to the avoidance of bankruptcy costs (see Diz and Whitman (2005)) and liquidation, we have that $\beta < \gamma$.

Let $1_{LIQ} = \{1$ if chapter 7 and $\tilde{\tau} \leq \overline{T}\}$ and $1_{RE} = \{1$ if chapter 11 and $\tilde{\tau} \leq \overline{T}\}$. The realization of chapter 7 versus chapter 11 depends on the value of the firm's assets if liquidated versus as a going concern at date $\overline{T}$, implying that these indicator functions are random variables. Note that $1_{LIQ} + 1_{RE} = 1_{\{\tilde{\tau} \leq \overline{T}\}}$.

Let $t > \tau$. Then, assuming α, β, γ are paid at time $\overline{T}$ and R, K are paid at time T:

$$
\begin{aligned}
R \; & \equiv \; E_{\tilde{\tau}}\{(\alpha 1_{LIQ} + \beta 1_{RE}) e^{-\int_T^{\overline{T}} r(s) ds} \big| \tilde{\tau} \leq \overline{T}\} \\
& < \; E_{\overline{T}}\{\gamma e^{-\int_T^{\overline{T}} r(s) ds} \big| \tilde{\tau} > \overline{T}\} \equiv K.
\end{aligned}
$$

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