A Note on Lando's Formula and Conditional Independence*

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Abstract

We extend Lando's formula for pricing credit risky derivatives to models where a firm's characteristics and its default point process need not be conditionally independent. This result is presented under a simple filtration expansion framework with basic probability techniques.

1 Introduction

Credit risk refers to the risk that the terms of a financial agreement may not be honored. As such, credit risk modeling studies the probability of such a failure or default, and the loss given default. An often used formula for pricing credit derivatives is contained in [28], where default is formulated as the first jump time of a Cox process (i.e., a doubly stochastic Poisson process) with a random intensity $(\lambda_t)_{t\geq 0}$ depending upon a finite dimensional stochastic process $(X_t)_{t\geq 0}$ characterizing the firm's economic condition. Lando shows that

$$P(\tau > t \mid (X_s)_{0\leq s\leq t}) = e^{-\int_0^t \lambda(X_s)\, ds}, \tag{1}$$

for any $(X_t)_{t\geq 0}$ that is right-continuous with left limits, where P is a martingale probability measure, E_1 is a unit exponential random variable independent of $(X_t)_{t\geq 0}$, and

$$\tau = \inf \left\{ t \mid \int_0^t \lambda(X_s)\, ds \geq E_1 \right\}. \tag{2}$$

Furthermore, the price of a defaultable, zero-coupon bond with a zero recovery rate is given by

$$1_{\{\tau>t\}} E_P \left[e^{-\int_t^T (r_s+\lambda_s)\, ds} \mid (X_s)_{0\leq s\leq t} \right], \tag{3}$$

where $(r_t)_{t\geq 0}$ is the default-free spot rate of interest.

This equation (3) is sometimes referred to as Lando's formula, and its importance is due to the fact that its form is the same as its default-free counterpart except for a modified spot rate

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$(r_s + \lambda_s)$. This simple modification enables the usage of numerous theorems available from the term structure of interest rate literature to price credit risky derivatives. The usefulness of this approach has led to various generalizations, see for example [2, 4, 5, 15, 17, 24, 30].

Our approach takes as given a (default time, filtration) pair $(\tau, \mathbb{F})$ where τ is a non-negative random variable, not necessarily an $\mathbb{F}$-stopping time, and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration on a probability space $(\Omega, \mathcal{F}, P)$. Next, we define $\mathbb{G}$ to be *any* filtration expansion of $\mathbb{F}$ such that τ is a $\mathbb{G}$-stopping time with

$$\mathcal{G}_t \cap \{t < \tau\} = \mathcal{F}_t \cap \{t < \tau\}. \tag{4}$$

Under this filtration expansion, we re-derive Lando's formula using an intensity based reduced-form model. We then extend Lando's formula for a class of structural models where the firm's asset value process is observed discretely. In this class, Lando's formula holds without the usual conditional independence assumption between the firm's default point process and state variables. And the proof uses only simple probability tools and the local deterministic property of the intensity process.

A word of caution here: this paper is by no means any attempt to review the vast existing literature on probability tools and models in credit risk. Rather, we aim at unwinding the relation of conditional independence assumption with the Lando's formula with the minimal probability background possible, and with a gentle touch on the filtration expansion theory. More mathematically sophisticated and curious minds are directed to [17, 24, 6], and to [5, 16, 36, 29] for comprehensive reviews.

An outline of our paper is as follows. Section 2 presents the mathematical setup. Lando's formula and its generalization are presented in Section 3. For completeness, the equivalence between two different definitions of an intensity process for any totally inaccessible stopping time τ for a given filtration $\mathbb{G}$ is provided in the Appendix.

2 The Mathematical Setup

2.1 Default Time τ, Filtration $(\mathcal{F}_t)_{t\geq 0}$, and Intensity of τ

Let us start with $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, a complete filtered probability space with $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ an arbitrary filtration that contains the null sets and is right-continuous.[1] Let τ be a non-negative random variable with $(N_t)_{t\geq 0}$ its associated point process, i.e. $N_t = 1_{\{\tau \leq t\}}$. For simplicity, we assume that $P(\tau = t) = 0$ for any t. Now define Z_t as the $\mathbb{F}$-optional projection of $1_{\{\tau > t\}}$, i.e.,

$$Z_t = E[1_{\{\tau > t\}} \mid \mathcal{F}_t] = P(\tau > t \mid \mathcal{F}_t). \tag{5}$$

Then, $(Z_t)_{t\geq 0}$ is clearly an $\mathbb{F}$-supermartingale.

In the context of credit risk modeling, τ represents the default time of a firm, with $\mathcal{F}_t$ being the information available to investors at time t, N_t the default indicator, and $Z_t = P(\tau > t \mid \mathcal{F}_t)$ the conditional survival probability at the time t given the information $\mathcal{F}_t$. Usually, the default time τ and the survival probability Z_t are the focus of economic analysis.

For any arbitrary non-negative random variable τ and a given filtration $\mathbb{F}$, τ may not be an $\mathbb{F}$ -stopping time. For example, consider a reduced-form model where $\mathcal{F}_t = \sigma(X_s, s \leq t)$ and τ is the first jump time of a Cox process with a random intensity depending upon X_t. Here τ is not an $\mathbb{F}$-stopping time. In this case, one can enlarge the filtration $\mathbb{F}$ to a bigger filtration $\mathbb{G}$ to make τ a $\mathbb{G}$-stopping time.

[1] These two conditions are sometimes referred to as "the usual conditions."

Moreover, given such a triple $(\tau, \mathbb{F}, \mathbb{G})$, $(N_t)_{t\geq 0}$ is a $\mathbb{G}$-submartingale. Therefore, by the Doob–Meyer decomposition, there exists a unique increasing $\mathbb{G}$-predictable process $(A_t^{\mathbb{G}})_{t\geq 0}$ with $A_0^{\mathbb{G}} = 0$ such that $(N_t - A_t^{\mathbb{G}})_{t\geq 0}$ is a $\mathbb{G}$-martingale. Here $(A_t^{\mathbb{G}})_{t\geq 0}$ is called the $\mathbb{G}$-*compensator* of τ.[2] Finally, if $(A_t^{\mathbb{G}})_{t\geq 0}$ is a.s. absolutely continuous with respect to Lebesgue measure, then the Radon–Nikodym derivative $(dA_t^{\mathbb{G}}/dt)_{t\geq 0}$ is called the *intensity process* $(\lambda_t)_{t\geq 0}$ of τ (see Brémaud [7, Chapter II, D7, T12, and T13]).

In the Appendix, we identify under proper integrability conditions the intensity process with its more intuitive counterpart, known as the Meyer's Laplacian approximation.

2.2 The Filtration Expansion $\mathbb{G}$ of $\mathbb{F}$

For a given pair $(\tau, \mathbb{F})$ for which τ is not an $\mathbb{F}$-stopping time, it is well known that there are many ways to expand the filtration $(\mathcal{F}_t)_{t\geq 0}$ in order to make τ a stopping time. Two standard approaches are the progressive filtration $\mathcal{G}$ and the minimal filtration $\mathbb{F}(\tau)$.

The progressive filtration $\mathcal{G}$ is [25, 37] obtained by defining $\mathcal{G}_\infty = \mathcal{F}_\infty \vee \sigma(\tau)$ with $\mathcal{F}_\infty = \vee_{t\geq 0}\mathcal{F}_t$, and

$$\mathcal{G}_t := \{B \in \mathcal{G}_\infty \mid \exists B_t \in \mathcal{F}_t \,:\, B \cap \{t < \tau\} = B_t \cap \{t < \tau\}\}. \tag{6}$$

Then $\mathcal{G} = (\mathcal{G}_t)_{t\geq 0}$ satisfies the usual conditions and it makes τ into a stopping time. Note that $\mathcal{F}_\infty \cap \{\tau \leq t\} \subset \mathcal{G}_t$, implying that after time τ, the progressively expanded filtration at time t includes all the information from the original filtration $\mathbb{F}$ up to ∞. In some economic settings, this could make the progressive filtration expansion unnatural for credit risk modeling.

The minimal filtration $\mathbb{F}(\tau)$ is obtained by enlarging $\mathbb{F}$ in the minimal way so as to include τ as a stopping time, i.e.

$$\mathcal{F}_t(\tau) = \mathcal{F}_t \vee \sigma(t \wedge \tau) = \mathcal{F}_t \vee \sigma(\{\tau \leq s\} \mid s \leq t). \tag{7}$$

This type of filtration expansion is explored, for example, in [24], [27], [17], [14], [19], and [6].

Note that $\mathcal{F}_t(\tau) \subset \mathcal{G}_t$ from expressions (6) and (7). And, both $\mathcal{G}_t$ and the right-continuous augmentation of $\mathcal{F}_t(\tau)$ coincide with $\mathcal{F}_t$ on $\{t < \tau\}$. Furthermore, many quantities associated with a default time, such as the intensity process, are well-defined only up to $t \leq \tau$ (see [24] for more discussions.)

In view of these observations, it suffices to specify the filtration expansion $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ to be *any* filtration expansion of $\mathbb{F}$ such that τ is a $\mathbb{G}$-stopping time with

$$\mathcal{G}_t \cap \{t < \tau\} = \mathcal{F}_t \cap \{t < \tau\}. \tag{8}$$

Given expression (8), we see $\mathbb{F}(\tau) \subset \mathbb{G}$. This is intuitive since at any give time t, one should know whether default happens or not by time t. Clearly both $\mathbb{F}(\tau)$ and $\mathcal{G}$ are special cases. (Interested readers are referred to [13], where this construction seems to first appear, for more discussions on this type of filtration).

Throughout the main section of this paper, unless otherwise specified, we consider only a non-negative random variable τ, the counting process of default $(N_t)_{t\geq 0}$, an arbitrary filtration $\mathbb{F}$, and its filtration enlargement $\mathbb{G}$ satisfying expression (8). This framework makes sense due to the following lemma (see [20]).

[2] Note that for purpose of exposition, the superscript $\mathbb{G}$ is used here for A to indicate the dependence of A on $\mathbb{G}$.

Lemma 1. *Compensators of τ under the expansion of class (8) are identical (up to τ). In particular, compensators of τ under its progressive expansion of $\mathbb{F}$ and under its minimal expansion of $\mathbb{F}$ are identical.*

3 A Generalized Lando's Formula

This section derives two extensions of Lando's formula using $\mathbb{G}$ from expression (8). The first extension is using the standard assumption of conditional independence in an intensity based model, and the second extension is without conditional independence for a class of structural models.

3.1 With Conditional Independence

This sections re-derives Lando's formula [28], assuming conditional independence, but under $\mathbb{G}$ of (8). First, as in [17, Proposition 3.1]:

Theorem 1. *For any $\mathcal{F}_T$-measurable integrable random variable X,*

$$E[X1_{\{\tau>T\}} \mid \mathcal{G}_t] = 1_{\{\tau>t\}} E\left[X\frac{Z_T}{Z_t} \mid \mathcal{F}_t\right] \quad \text{for } t < T. \tag{9}$$

Proof. Since X is $\mathcal{F}_T$ measurable, by properties of conditional expectations and the Bayes lemma (see for instance [17]),

$$\begin{aligned} 1_{\{\tau>t\}} E[X\frac{Z_T}{Z_t} \mid \mathcal{F}_t] &= 1_{\{\tau>t\}} E\left[X\frac{P(\tau>T \mid \mathcal{F}_T)}{P(\tau>t \mid \mathcal{F}_t)} \mid \mathcal{F}_t\right] \\ &= 1_{\{\tau>t\}} \frac{E[E[X1_{\{\tau>T\}} \mid \mathcal{F}_T] \mid \mathcal{F}_t]}{E[1_{\{\tau>t\}} \mid \mathcal{F}_t]} \\ &= 1_{\{\tau>t\}} \frac{E[X1_{\{\tau>T\}} \mid \mathcal{F}_t]}{E[1_{\{\tau>t\}} \mid \mathcal{F}_t]} \\ &= E[X1_{\{\tau>T\}} \mid \mathcal{G}_t]. \end{aligned}$$

□

Next, letting τ be the first jump time of a Cox process so that expression (1) holds, setting $X \equiv 1$ and $\mathcal{F}_t = \sigma(X_s, s \le t)$, we get[3]

$$\begin{aligned} P(\tau > T \mid \mathcal{G}_t) &= 1_{\{\tau>t\}} E\left[\frac{P(\tau>T \mid \mathcal{F}_T)}{P(\tau>t \mid \mathcal{F}_t)} \mid \mathcal{F}_t\right] \\ &= 1_{\{\tau>t\}} E\left[\frac{e^{-\int_0^T \lambda(X_s)\,ds}}{e^{-\int_0^t \lambda(X_s)\,ds}} \mid \mathcal{F}_t\right] \\ &= 1_{\{\tau>t\}} E[e^{-\int_t^T \lambda(X_s)\,ds} \mid \mathcal{F}_t]. \end{aligned}$$

Last, letting $(r_t)_{t\ge0}$ be the $\mathbb{F}$-adapted default-free spot rate of interest, then under suitable integrability conditions as in [28, Proposition 3.1], and by arguments as in Proposition 1 and in [28], we obtain the following extensions of Lando's formulas [28] under $\mathbb{G}$.

[3]This is the essence of Proposition 3.1 in Lando [28].

Corollary 1. *Assume the conditional independence in the sense of expression (1). Assume that* X_T *is* $\mathcal{F}_T$*-measurable, and* Y *and* V *are* $\mathbb{F}$*-adapted where* $\mathcal{F}_t = \sigma(X_s, s \le t)$*, so that* $E\left[e^{-\int_t^T r_s\,ds}|X|\right]$, $E\left[\int_t^T e^{-\int_t^s r_u\,du}|Y_s|\,ds\right]$ *and* $E\left[\int_t^T e^{-\int_t^s (r_u+\lambda_u)\,du}|V_s\lambda_s|\,ds\right]$ *are all finite. Then,*

$$E\left[e^{-\int_t^T r_s\,ds} X_T 1_{\{\tau>T\}} \mid \mathcal{G}_t\right] = 1_{\{\tau>t\}} E\left[e^{-\int_t^T (r_s+\lambda_s)\,ds} X_T \mid \mathcal{F}_t\right],$$

$$E\left(\int_t^T Y_s 1_{\{\tau>s\}} e^{-\int_t^s r_u\,du}\,ds \mid \mathcal{G}_t\right) = 1_{\{\tau>t\}} E\left[\int_t^T Y_s e^{-\int_t^s (r_u+\lambda_u)\,du}\,ds \mid \mathcal{F}_t\right],$$

and

$$E\left[e^{-\int_t^\tau r_s\,ds} V_\tau \mid \mathcal{G}_t\right] = 1_{\{\tau>t\}} E\left[\int_t^T V_s \lambda_s e^{-\int_t^s (r_u+\lambda_u)\,du}\,ds \mid \mathcal{F}_t\right].$$

3.2 Without Conditional Independence

In this section we derive Lando's formula for a naturally occurring class of filtrations $\mathbb{F}$ that do not *a priori* assume conditional independence. As an important special case, this class includes the situation where a firm's continuous time asset value process is only observed at discrete time intervals.

Theorem 2 (Generalized Lando's Formula under Discrete Observations)**.** *Let* $(X_t)_{t\ge0}$ *be a one-dimensional, time-homogeneous Markov process with a continuous sample path. Let* $(T_n)_{n\in\mathbb{N}}$ *be a deterministic and strictly increasing sequence.*

If $\tau = \inf\{t > 0 \mid X_t \in D\}$ *for a Borel subset of the state space* D *has a continuous probability density function* $f(x,t)$ *where* $f(x,t)dt = P_x(\tau \in dt)$*, and if* $\tau > t$ *and* $T_n \le t < T_{n+1}$*, then we have*

$$e^{-\int_t^{T_{n+1}} \lambda_s\,ds} 1_{\{\tau>t\}} = P(\tau > T_{n+1} \mid \mathcal{G}_t), \tag{10}$$

where $\mathcal{F}_t = \{\sigma(X_{T_i}), T_i < t\}$, $\mathbb{G} = (\mathcal{G}_t)_{t\ge0}$ *is any enlarged filtration of* $\mathbb{F}$ *from expression (8), and* $\lambda_s = \frac{f(x - X_{T_n}, s-T_n)}{P_{X_{T_n}}(\tau > s-T_n)}$ *for* $s \in [t, T_{n+1})$.

The proof relies on the following result (see [33, Pages 34-35]).

Lemma 2. *Let* $(X_t)_{t\ge0}$ *be a one-dimensional, time-homogeneous Markov process with a continuous sample path. Let* $\mathcal{F}_t = \{\sigma(X_{T_i}), T_i < t\}$ *be a deterministic delayed filtration.*

If $\tau = \inf\{t > 0 \mid X_t \in D\}$ *for a Borel subset of the state space* D *has a continuous density function* $f(x,t)$ *where* $f(x,t)dt = P_x(\tau \in dt)$*, then on* $\tau > t$ *and* $T_n \le t < T_{n+1}$*, the* $\mathbb{F}(\tau)$*- stopping time* τ *has a default intensity* $\frac{f(x - X_{T_n}, t-T_n)}{P_{X_{T_n}}(\tau > t-T_n)}$.

Proof. Clearly, for $s = t = T_n, \lambda_s = 0$, so without loss of generality we consider only $T_n < t < T_{n+1}$ and $\mathcal{F}_t = \{\sigma(X_{T_i}), T_i < t\}$. In this case, $\lambda_s = \frac{f(x - X_{T_n}, s-T_n)}{P_{X_{T_n}}(\tau > s-T_n)}$ for $\tau > t$ and $s \in [t, T_{n+1})$. Thus denoting $Z_s = P_{X_{T_n}}(\tau > s - T_n)$, we see

$$\begin{aligned}
e^{-\int_t^{T_{n+1}} \lambda_s\, ds} 1_{\{\tau>t\}} &= e^{\int_t^{T_{n+1}} \frac{Z'(s)}{Z(s)}\, ds} 1_{\{\tau>t\}} \\
&= e^{\int_t^{T_{n+1}} \frac{dZ(s)}{Z(s)}} 1_{\{\tau>t\}} \\
&= \frac{P_{X_{T_n}}(\tau > T_{n+1} - T_n)}{P_{X_{T_n}}(\tau > t - T_n)} 1_{\{\tau>t\}} \\
&= \frac{1_{\{\tau>T_n\}} P_{X_{T_n}}(\tau > T_{n+1} - T_n)}{1_{\{\tau>T_n\}} P_{X_{T_n}}(\tau > t - T_n)} 1_{\{\tau>t\}} \\
&= \frac{P(\tau > T_{n+1} \mid \mathcal{F}_{T_n})}{P(\tau > t \mid \mathcal{F}_{T_n})} 1_{\{\tau>t\}} \\
&= P[\tau > T_{n+1} \mid \mathcal{G}_t].
\end{aligned}$$

The Markov property is applied to get the fifth equation, while the last equation is due to the Bayes formula. □

Discussion: A comparison with Lando [28]. Recall that in an intensity based reduced-form model as in [28], the default time represents the first jump time of a Cox process, with intensity $\lambda = (\lambda(X_t))_{t\geq 0}$. And expression (1) is derived using the *conditional independence* assumption between $(X_t)_{t\geq 0}$ and E_1, which also yields the following relation [28, Page 104, Eq. (3.4)]:

$$E\left[1_{\{\tau>T\}} \mid \sigma((X_s)_{0\leq s\leq T}) \vee \sigma((N_s)_{0\leq s\leq t})\right] = 1_{\{\tau>t\}} \exp\left(-\int_t^T \lambda_s\, ds\right). \quad (11)$$

The filtration in expression (11) involves $\sigma((X_s)_{0\leq s\leq T})$, the information up to time T.

In contrast, our result holds both with and without the *conditional independence* assumption. Moreover, in the latter case where the filtration is generated by the discrete observations of X, we see that

$$P(\tau > T_{n+1} \mid \mathcal{G}_t) = E[1_{\{\tau>T_{n+1}\}} \mid \mathcal{G}_t] = 1_{\{\tau>t\}} e^{-\int_t^{T_{n+1}} \lambda_s\, ds} \quad (12)$$

where $\mathcal{G}_t$ takes the form of $\sigma((X_{T_i}, T_i \leq t) \vee \sigma((N_s)_{0\leq s\leq t})$ according to Theorem 2. Note also that τ in expression (12) is not the first jump time of a Cox process. Instead, $\tau := \inf\{t > 0, X_t \in D\}$ for some Borel subset of the state space. This later formulation is the standard default time definition used in structural models for credit risk.

Finally, we note that [24] proposes an idea similar to that in Theorem 2.

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4 Appendix: The $\mathbb{G}$-intensity and Meyer's Laplacian Approximation

In this Appendix, we identify the intensity process with the Meyer's Laplacian approximation, under proper integrability conditions.

It is important to emphasize that here we assume $\mathbb{G}$ to be any appropriate filtration expansion of $\mathbb{F}$, and not necessarily of expression (8).

4.1 $(\lambda_t)_{t\geq 0}$ from Meyer's Laplacian Approximation

Duffie and Lando [14] showed that if the hypothesis for using dominating convergence is satisfied as required by Aven's lemma [3], then the $\mathbb{G}$-intensity can be calculated via Meyer's Laplacian approximation.[4] That is, let

$$\bar{\lambda}_t = \lim_{h\downarrow 0} \frac{1}{h} P(t < \tau \leq t+h \mid \mathcal{G}_t). \tag{13}$$

so that $\bar{\lambda}_t$ is the instantaneous likelihood of default at time t given $\mathcal{G}_t$. Then, under Aven's lemma, it identifies with λ_t, the Radon–Nikodym derivative of the $\mathbb{G}$-compensator of τ.

The converse is true: if the Radon–Nikodym derivative of the $\mathbb{G}$-compensator satisfies proper integrability conditions, then it identifies with the Laplacian approximation $\bar{\lambda}_t$. We provide several such conditions below.

For simplicity, we will assume without loss of generality that $0 < Z_t < 1$ for any $t > 0$, and impose the following two assumptions.

Assumption A1. The $\mathbb{G}$-stopping time τ has an intensity process $(\lambda_t)_{t\geq 0}$ with right-continuous sample paths.

Assumption A2. The limit $\bar{\lambda}_t = \lim_{h\downarrow 0} \frac{1}{h} P(t < \tau \leq t+h \mid \mathcal{G}_t)$ exists almost surely and is a.s. right-continuous.

Proposition 1. *Given any $\mathbb{G}$-stopping time τ, together with assumptions **A1** and **A2**. If for any given $t > 0$ there exists an $h_0(t)$ so that $(\frac{1}{h}\int_t^{t+h} \lambda_s)_{h>0}$ is uniformly integrable for $h \in (0, h_0]$, then $(\lambda_t)_{t\geq 0}$ is indistinguishable from $(\bar{\lambda}_t)_{t\geq 0}$.*

Proof. By the right-continuity of λ and $\bar{\lambda}$, it suffices to show that for any $t > 0$, $\lambda_t = \bar{\lambda}_t$ a.s.

First, take the Doob–Meyer decomposition of $(N_t)_{t\geq 0}$ so that $N_t = M_t + A_t^{\mathbb{G}}$, where $(M_t)_{t\geq 0}$ is an $\mathbb{G}$-martingale and $(A_t^{\mathbb{G}})_{t\geq 0}$ is an increasing $\mathbb{G}$-predictable process. Since $A_t^{\mathbb{G}} = \int_0^t \lambda_s ds$, then for any fixed t,

$$\begin{aligned}\frac{1}{h} P(t < \tau \leq t+h \mid \mathcal{G}_t) \quad &= \frac{1}{h} E\left[N_{t+h} - N_t \mid \mathcal{G}_t\right] = \frac{1}{h} E\left[A_{t+h}^{\mathbb{G}} - A_t^{\mathbb{G}} \mid \mathcal{G}_t\right] \\ &= \frac{1}{h} E\left[\int_t^{t+h} \lambda_s ds \mid \mathcal{G}_t\right].\end{aligned}$$

Next, the right-continuity of λ implies $\lim_{h\downarrow 0} \frac{1}{h}\int_t^{t+h} \lambda_s = \lambda_t$. This together with the uniform integrability shows that $(\frac{1}{h}\int_t^{t+h} \lambda_s)_{h>0}$ converges to λ_t in L^1. Hence

$$\left(E\left[\frac{1}{h}\int_t^{t+h} \lambda_s ds \mid \mathcal{G}_t\right]\right)_{h>0} \to \lambda_t \text{ in } L^1.$$

Therefore,

$$\bar{\lambda}_t \quad = \quad \lim_{h\downarrow 0} \frac{1}{h} P(t < \tau \leq t+h \mid \mathcal{G}_t) = \lim_{h\downarrow 0} E\left[\frac{1}{h}\int_t^{t+h} \lambda_s ds \mid \mathcal{G}_t\right] = \lambda_t. \quad \square$$

There are various technical conditions sufficient for obtaining the uniform integrability condition, see [35]. Here are a few possibilities.

[4] The link between hazard rate (or default intensity) and survival probability has long before been studied in the survival analysis literature (see e.g., [26] for an overview) but the role of the filtration has not been examined.

Corollary 2. *Given any $\mathbb{G}$-stopping time τ, together with assumptions **A1** and **A2**. If for any given $t > 0$ there exists $p = p(t) > 1$ and $h_0 = h_0(t) > 0$ so that $\frac{1}{h}\int_t^{t+h} E[\lambda_s^p]$ is bounded for $h \in (0, h_0]$, then λ is indistinguishable from $\bar{\lambda}$.*

Proof. Let q be the number conjugate to p so that $\frac{1}{p} + \frac{1}{q} = 1$, then for h sufficiently small,

$$\begin{aligned} E\left[\left(\frac{1}{h}\int_t^{t+h} \lambda_s\right)^p\right] &\leq E\left[\left(\frac{1}{h}\sqrt[p]{\int_t^{t+h} \lambda_s^p}\sqrt[q]{\int_t^{t+h} 1}\right)^p\right] \\ &= E\left[\frac{1}{h^p}\int_t^{t+h} h^{p-1}\lambda_s^p\right] = \frac{1}{h}\int_t^{t+h} E[\lambda_s^p]\,. \end{aligned}$$

So $\left(\frac{1}{h}\int_t^{t+h} \lambda_s\right)_{h>0}$ is uniformly integrable, and statement now follows from Proposition 1. □

Corollary 3. *Given a $\mathbb{G}$-stopping time τ, together with assumptions **A1** and **A2**. If $E\{\lambda_t^p\}$ is bounded on any finite interval, then λ is indistinguishable from $\bar{\lambda}$.*

Remark 1. *To our best knowledge, the only other existing reference concerning the equivalence of an intensity process of a stopping time and its Meyer's Laplacian approximation is Proposition 5.10 of [18]. We give a more general characterization by specifying weaker conditions of λ_t.*