

Optimal execution with multiplicative price impact*

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Abstract. We consider the so-called “optimal execution problem” in algorithmic trading, which is the problem faced by an investor who has a large number of stock shares to sell over a given time horizon and whose actions have impact on the stock price. In particular, we develop and study a price model that presents the stochastic dynamics of a geometric Brownian motion and incorporates a log-linear effect of the investor’s transactions. We then formulate the optimal execution problem as a two-dimensional degenerate singular stochastic control problem. Using both analytic and probabilistic techniques, we establish simple conditions for the market to allow for no price manipulation and we develop a detailed characterisation of the value function and the optimal strategy. In particular, we derive an explicit solution to the problem if the time horizon is infinite. Interesting features of the problem’s solution include the facts that (a) the value function may be discontinuous as a function of the time horizon and (b) an optimal strategy may not exist even when the value function is finite.

Key words. optimal execution problem, multiplicative price impact, singular stochastic control

AMS subject classifications. 93E20, 91G80, 49L20

1. Introduction. We consider an investor who has a large number of stock shares to sell within a given time frame. Rapid selling of the stock may depress the stock price, while slicing the big order into many smaller blocks of orders to be executed sequentially over time may take too long to realise. Such an investor is therefore faced with the problem of how to slice the order, when to trade and at what price, etc. This problem, known as the “optimal execution problem” in algorithmic trading, is concerned with finding a trading strategy that maximises an appropriate objective function. A key issue of the problem is concerned with modelling the price impact of stock transactions.

The study of the optimal execution problem was initiated by Bertsimas and Lo [8] who analysed a discrete random walk model and by Almgren and Chriss [5, 6] and Almgren [4] who considered continuous time Bachelier-type models with additive price impact. Since then, the area has attracted considerable interest; an incomplete list of notable contributions in the mathematics literature includes Huberman and Stanzl [21], He and Mamaysky [20], Obizhaeva and Wang [25], Almgren and Lorenz [7], Engle and Ferstenberg [13], Schied and Schöneborn [29], Alfonsi, Fruth and Schied [1, 2], Schied, Schöneborn and Tehranchi [30], Predoiu, Shaikhet and Shreve [27] and Løkka [23].

Modelling stock prices by an arithmetic Brownian motion / random walk with additive impact of large stock sales is a common feature in the references on the optimal execution problem discussed above. An intriguing consequence of this modelling approach is that optimal strategies turn out to be more or less static or deterministic. Such strategies may lead to predictable trading patterns, which can give rise to market manipulation with techniques such as predatory trading (to this end, see the game formulations studied by Schied and

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Schöneborn [28] and Moallemi, Park and Van Roy [24]). Recent work by Schied, Schöneborn and Tehranchi [30], Gatheral and Schied [17] and Predoiu, Shaikhet and Shreve [27] has revealed that such deterministic optimal strategies can be recovered by a simple argument involving an integration-by-parts calculation and an appropriate Euler-Lagrange equation, establishing an effective equivalence between minimising costs and minimising the price impact of trading strategies.

Beyond the context of Bachelier-type models, Gatheral and Schied [17] studied a continuous time Black and Scholes-type model with additive price impact. Discrete time models with multiplicative price impact have been considered by Bertsimas and Lo [8] and Bertsimas, Lo and Hummel [9]. Also, Forsyth, Kennedy, Tse and Windcliff [14, 15] proposed a continuous time Black and Scholes-type model with multiplicative price impact and derived its Hamilton-Jacobi-Bellman (HJB) equation using heuristic arguments, which they studied by means of numerical techniques. In these references, it is argued that such models are more natural than ones with additive price impact because, e.g., they do not allow for strictly negative prices with non-zero probability.

In this paper, we study the optimal execution problem in the context of a continuous time model with multiplicative price impact. To the best of our knowledge, this model is the very first one in the continuous time optimal execution literature involving singular control rather than absolutely continuous control: this setting does not restrict stock transactions to be realised at a rate over time; instead, it allows for block sales of stock. The objective of the paper is to exhaustively study the model's analytical properties. The development of further realistic and applicable models can be motivated by the one we study here (see Remark 2 for such a generalisation).

In particular, we consider an investor who holds $Y_t \geq 0$ shares of stock at time t , not including any transactions made at t . The investor can buy or sell any amount of shares at any time, but short-selling is not allowed. We denote by ξ_t^s (resp., ξ_t^b) the total amount of shares the investor has sold (resp., bought) up to time t , so that

$$Y_t = y - \xi_t^s + \xi_t^b,$$

where $y \geq 0$ is the number of shares held by the investor at time 0.

We assume that, in the absence of any transactions, stock prices follow a geometric Brownian motion. Also, we assume that (a) the price impact of small transactions is proportional to the stock price at which they are executed as well as proportional to their size, and (b) the price impact of a large transaction is the same as that of any number of smaller transactions of the same total size that are executed at the same time. In §2, we show that such requirements give rise to the stock price dynamics

$$dX_t = \mu X_t dt - \lambda X_t \circ_\lambda d\xi_t^s + \kappa X_t \circ_\kappa d\xi_t^b + \sigma X_t dW_t,$$

where $\lambda, \kappa > 0$ are constants and the operators $\circ_\lambda, \circ_\kappa$ are defined by (2.4)–(2.5) below. Effectively, this is a model with multiplicative price impact: the impact of a transaction is additive to the logarithm of the stock price (see (2.6) below). There are several generalisations of these dynamics that exhibit resilience, namely, capture the possibility for the effect of transactions on the stock price to fade over time (see Remark 2).

The investor has a horizon $\bar{T} \in (0, \infty]$, by which time, she exits the market by clearing all her shares. The investor's objective is to maximise the performance criterion

$$\mathbb{E} \left[\int_{[0, \bar{T}] \cap \mathbb{R}_+} e^{-\delta t} \left[(X_t - C_s) \circ_{\lambda} d\xi_t^s - (X_t + C_b) \circ_{\kappa} d\xi_t^b \right] \right]$$

over all admissible strategies (ξ^s, ξ^b) . Here, the constant $\delta \geq 0$ reflects the investor's impatience, while the constants $C_s, C_b \geq 0$ provide for a bid-ask spread or for proportional transaction costs. The choice $\delta = 0$ is the most natural one if the time horizon \bar{T} is very short. We allow for choices $\delta > 0$ because these might be appropriate for execution problems lasting several days (see Lebedeva, Maug and Schneider [22] for real-world examples of such executions) and are essential for a non-trivial solution if $\bar{T} = \infty$. Also, strictly positive values of C_s, C_b can arise from the existence of a bid-ask spread. Indeed, if we interpret X_t as the middle stock price at time t , then we can view $X_t - C_s$ (resp., $X_t + C_b$) as the bid (resp., ask) price of the stock at time t . Such a modelling context has been considered in the literature, e.g., by Cont and de Larrard [10] who, based on empirical evidence, assume that the bid-ask spread is equal to one tick.

The performance criterion we have adopted is the expected revenue one featuring in the models studied, e.g., by Bertsimas and Lo [8] and Gatheral [16]. Other choices of performance criteria that have been considered in the literature include the mean-variance one in Almgren and Chriss [5, 6], the expected utility one in Schied and Schöneborn [28] and the mean-quadratic variation one in Forsyth, Kennedy, Tse and Windcliff [14]. Such alternative performance indices give rise to several variants of the model we study that could be the subject of future research. It is worth noting that Gatheral and Schied [18] have argued that a risk-neutral expected revenue or cost optimization objective is a reasonable choice, especially in contexts where market regularity conditions should be independent of investors preferences.

Mathematically, the optimisation problem above takes the form of a singular stochastic control problem. Its Hamilton-Jacobi-Bellman (HJB) equation is a two-dimensional degenerate parabolic (if $\bar{T} < \infty$) or elliptic (if $\bar{T} = \infty$) PDE with state-dependent gradient constraints. Although the literature of singular stochastic control is rich and long, we are unaware of any results that characterise the value function or the optimal strategies in a context similar to the one we consider here; models that are closest to the one we analyse have been studied by Shreve and Soner [31, 32], Davis and Norman [12], Zhu [33], Ocone and Weerasinghe [26] and Dai and Yi [11].

Our analysis involves probabilistic as well as analytic techniques. It turns out that the problem we study is associated with a rather rich family of optimal strategies. A brief summary of our main results is as follows.

First, we show that the value function is finite if and only if both of the following conditions hold true:

$$(1.1) \quad \text{(a) } \kappa \leq \lambda \quad \text{and} \quad \text{(b) if } \mu > \delta, \text{ then } \bar{T} \leq \frac{1}{\mu - \delta} \ln \frac{\lambda}{\kappa}.$$

In the case when $\bar{T} < \infty$, we prove a verification theorem (Proposition 3) that relates appropriate super-solutions to the problem's HJB equation to the problem's value function. Beyond

its independent interest, we combine this proposition with the preliminary results we obtain in §3 by means of probabilistic techniques to derive a detailed characterisation of the value function and the optimal strategies in Proposition 4. Interesting control theoretic conclusions reached there include the facts that, if $\mu > \delta$, then the solution to the control problem is discontinuous as a function of the time horizon \bar{T} , while, if $\mu < \delta$, then buying is part of the optimal tactics.

If $\bar{T} = \infty$, then we derive the solution to the problem in an explicit form (see Proposition 5). An interesting feature of this solution is that an optimal strategy may not exist even when the value function is finite (see Proposition 5.(II)). If the value function is finite and the optimal strategy exists, then the optimal strategy can be described informally as follows (see also Figure 1). If the stock price is below a critical level F_\circ , then it is optimal to take no action. If the stock price at time 0 is above F_\circ , then it is optimal to either sell all available shares immediately or liquidate an amount that would cause the stock price to drop to F_\circ and then keep on selling until all shares are exhausted by just preventing the stock price to rise above F_\circ .

The mathematical results that we derive provide several economic insights into the model. First, if $\kappa > \lambda$, then the value function is infinite because the market allows for arbitrage opportunities (see Remark 1). If $\kappa \leq \lambda$ and $\mu > \delta$, then the model allows for price manipulation in the sense of Huberman and Stanzl [21], while, the inequalities $\kappa \leq \lambda$ and $\mu \leq \delta$ are sufficient for excluding price manipulation (see Remark 3). Furthermore, these inequalities are necessary and sufficient for excluding transaction-triggered price manipulation in the sense of Alfonsi, Schied and Slynko [3]. Indeed, if $\kappa \leq \lambda$ and $\mu \leq \delta$, then the optimal tactics do not involve buying of shares, while, if $\kappa > \lambda$ or $\mu > \delta$, then buying of shares is part of the optimal activity. It is worth noting that, in the multiplicative model that we study, it is possible to have asymmetric price impact (namely, $\kappa < \lambda$) without price manipulation. This is different from the models with additive price impact studied by Huberman and Stanzl [21] and Gatheral [16], who showed that permanent price impact must be linear and symmetric to exclude price manipulation.

2. The market model and the control problem. We fix a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual conditions and carrying a standard (\mathcal{F}_t) -Brownian motion W .

We denote by Y_t the total number of shares held by the investor at time t . Also, we denote by ξ_t^s (resp., ξ_t^b) the total number of shares that the investor has sold (resp., bought) up to time t , so that $Y_t = y - \xi_t^s + \xi_t^b$, where $y \geq 0$ is the number of shares held by the investor at time 0. We assume that ξ^s and ξ^b are (\mathcal{F}_t) -adapted increasing càglàd processes such that $\xi_0^s = \xi_0^b = 0$. Also, we assume that a primary aim of the investor is to liquidate all share holdings by a time horizon $\bar{T} \in (0, \infty]$. We therefore consider only trading strategies (ξ^s, ξ^b) such that

$$(2.1) \quad Y_{\bar{T}+} = 0, \text{ if } \bar{T} < \infty, \quad \text{and} \quad \lim_{T \rightarrow \infty} Y_T = 0, \text{ if } \bar{T} = \infty.$$

In the absence of any transaction from the investor, we model the stock price by the geometric Brownian motion X^0 given by

$$(2.2) \quad dX_t^0 = \mu X_t^0 dt + \sigma X_t^0 dW_t, \quad X_0^0 = x > 0,$$

for some constants μ and $\sigma \neq 0$. We assume that small transactions made by the investor affect the share price proportionally to its value. In particular, if the investor sells (resp., buys) a small amount $\varepsilon > 0$ of shares at time t , then the share price exhibits a jump of size

$$\Delta X_t = X_{t+} - X_t = -\lambda\varepsilon X_t \quad (\text{resp.}, \Delta X_t = X_{t+} - X_t = \kappa\varepsilon X_t),$$

for some constants $\kappa, \lambda > 0$, where we have assumed that X is càglàd. In this context, a small sale (resp., buy) of size $\varepsilon > 0$ is associated with the expressions

$$X_{t+} = (1 - \lambda\varepsilon)X_t \simeq e^{-\lambda\varepsilon}X_t \quad (\text{resp.}, X_{t+} = (1 + \kappa\varepsilon)X_t \simeq e^{\kappa\varepsilon}X_t).$$

If we view the sale of $\Delta\xi_t^s$ shares as N individual sales of $\varepsilon = \Delta\xi_t^s/N$ shares each, then, for N large enough, we obtain

$$X_{t+} = e^{-\lambda N\varepsilon}X_t = e^{-\lambda\Delta\xi_t^s}X_t.$$

Similarly, we can see that buying $\Delta\xi_t^b$ shares is associated with the jump $X_{t+} = e^{\kappa\Delta\xi_t^b}X_t$.

In view of the above considerations, we model the stock price dynamics by the stochastic equation

$$(2.3) \quad dX_t = \mu X_t dt - \lambda X_t \circ_\lambda d\xi_t^s + \kappa X_t \circ_\kappa d\xi_t^b + \sigma X_t dW_t,$$

where

$$(2.4) \quad X_t \circ_\lambda d\xi_t^s = X_t d(\xi^s)_t^c + \frac{1}{\lambda} X_t \left[1 - e^{-\lambda\Delta\xi_t^s} \right] = X_t d(\xi^s)_t^c + X_t \int_0^{\Delta\xi_t^s} e^{-\lambda u} du$$

and

$$(2.5) \quad X_t \circ_\kappa d\xi_t^b = X_t d(\xi^b)_t^c + \frac{1}{\kappa} X_t \left[e^{\kappa\Delta\xi_t^b} - 1 \right] = X_t d(\xi^b)_t^c + X_t \int_0^{\Delta\xi_t^b} e^{\kappa u} du,$$

where the process $(\xi^s)^c$ (resp., $(\xi^b)^c$) is the continuous part of the process ξ^s (resp., ξ^b). Using Itô's formula, we can verify that the solution to (2.3) is given by

$$(2.6) \quad X_t = x \exp \left(\left(\mu - \frac{1}{2}\sigma^2 \right) t - \lambda\xi_t^s + \kappa\xi_t^b + \sigma W_t \right) = X_t^0 \exp \left(-\lambda\xi_t^s + \kappa\xi_t^b \right)$$

where X^0 is the solution to (2.2).

If we consider the sale of $\Delta\xi_t^s$ shares at time t as equivalent to the sale of N packets of shares of small size $\varepsilon = \Delta\xi_t^s/N$, then we can see that such a sale should result in a revenue of

$$\sum_{j=0}^{N-1} e^{-\lambda j\varepsilon} X_t \varepsilon \simeq \int_0^{\Delta\xi_t^s} X_t e^{-\lambda u} du = \frac{1}{\lambda} X_t \left[1 - e^{-\lambda\Delta\xi_t^s} \right].$$

In view of this observation and a similar one concerning the buying of $\Delta\xi_t^b$ shares at time t , we associate the performance criterion

$$(2.7) \quad I_{\bar{T},x,y}(\xi^s, \xi^b) = \begin{cases} J_{\bar{T},x,y}(\xi^s, \xi^b), & \text{if } \bar{T} < \infty, \\ \limsup_{T \rightarrow \infty} J_{T,x,y}(\xi^s, \xi^b), & \text{if } \bar{T} = \infty, \end{cases}$$

with each liquidation strategy (ξ^s, ξ^b) , where $J_{T,x,y}(\xi^s, \xi^b)$ is defined by

$$J_{T,x,y}(\xi^s, \xi^b) = \mathbb{E} \left[\int_{[0,T]} e^{-\delta t} \left[X_t \circ_{\lambda} d\xi_t^s - X_t \circ_{\kappa} d\xi_t^b - C_s d\xi_t^s - C_b d\xi_t^b \right] \right],$$

for $(T, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}_+^1$. Here, the discounting rate $\delta \geq 0$ reflects the investor's "impatience", while the constants $C_s, C_b \geq 0$ may account for a bid-ask spread or provide for proportional transaction costs. In what follows, we will often use the more compact expression

$$(2.8) \quad J_{T,x,y}(\xi^s, \xi^b) = \mathbb{E} \left[\int_{[0,T]} e^{-\delta t} \left[(X_t - C_s) \circ_{\lambda} d\xi_t^s - (X_t + C_b) \circ_{\kappa} d\xi_t^b \right] \right],$$

which is consistent with (2.4)–(2.5).

The investor's objective is to maximise $I_{\bar{T},x,y}(\xi^s, \xi^b)$ over all liquidation strategies (ξ^s, ξ^b) . Accordingly, we define the problem's value function v by

$$(2.9) \quad v(\bar{T}, x, y) = \sup_{(\xi^s, \xi^b) \in \mathcal{A}_{\bar{T},y}} I_{\bar{T},x,y}(\xi^s, \xi^b),$$

where $\mathcal{A}_{\bar{T},y}$ is the family of all admissible strategies, which is introduced by the following definition.

Definition 1. Given a time horizon $\bar{T} \in (0, \infty]$ and an initial holding of $y \geq 0$ shares, the family $\mathcal{A}_{\bar{T},y}$ of all admissible liquidation strategies is the set of all pairs (ξ^s, ξ^b) composed by (\mathcal{F}_t) -adapted increasing càglàd processes ξ^s and ξ^b such that $\xi_0^s = \xi_0^b = 0$,

$$(2.10) \quad Y_t = y - \xi_t^s + \xi_t^b \geq 0 \quad \text{for all } t \in [0, \bar{T}] \cap \mathbb{R}_+,$$

$$(2.11) \quad \mathbb{E} \left[e^{4\kappa \xi_{T+}^b} \right] < \infty \quad \text{for all } T \in \mathbb{R}_+,$$

and (2.1) holds true. We denote by $\mathcal{A}_{\bar{T},y}^s$ the family of all processes ξ^s such that $(\xi^s, 0) \in \mathcal{A}_{\bar{T},y}$.

The integrability condition (2.11) is quite general and ensures that the optimisation problem is well-posed (see Lemma 1). Also, the inequality $Y_t \geq 0$ for all $t \in [0, T]$ reflects the idea that the possibility of short-selling is not permitted.

The model that we have developed allows for the investor to simultaneously buy and sell shares, which is consistent with practice. For instance, the investor can simultaneously sell an amount $\Delta \xi_t^s$ and buy an amount $\Delta \xi_t^b$ of shares at time t , which would result in the share price moving from X_t to $e^{-\lambda \Delta \xi_t^s + \kappa \Delta \xi_t^b} X_t$. However, such simultaneous actions are effectively ruled out by the fact that they are sub-optimal because they result in a net loss (see Lemma 1).

In the next assumption we summarise the possible values that the various constants we have considered may take.

Assumption 1. $\mu, \sigma \neq 0$, $\delta \geq 0$ and $\kappa, \lambda > 0$ are constants such that $\kappa \leq \lambda$.

¹Throughout the paper, we use the notation $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_+^* = (0, \infty)$.

Remark 1. If we allow for the inequality $\kappa > \lambda$ to be true, then the market may offer arbitrage opportunities and the value function v may be identically equal to ∞ . To see this claim, suppose that $\kappa > \lambda$, and suppose that the investor buys $\varrho > 0$ shares at time 0 and then immediately sells them. These transactions result in the share price jumping from x up to $xe^{\kappa\varrho}$ and then down to $xe^{\kappa\varrho}e^{-\lambda\varrho} > x$. The investor's revenue from these actions is

$$\begin{aligned} & -\frac{1}{\kappa}x[e^{\kappa\varrho} - 1] + \frac{1}{\lambda}xe^{\kappa\varrho}[1 - e^{-\lambda\varrho}] - (C_s + C_b)\varrho \\ & = \frac{1}{\lambda}x \left[\frac{\kappa - \lambda}{\kappa} [e^{\kappa\varrho} - 1] - [e^{(\kappa-\lambda)\varrho} - 1] \right] - (C_s + C_b)\varrho =: q(\varrho) \end{aligned}$$

If $C_s = C_b = 0$, then we can see that $q(\varrho) > 0$ for all $\varrho > 0$ because $q(0) = 0$ and $q'(\varrho) > 0$ for all $\varrho > 0$. On the other hand, if $C_s + C_b > 0$, then $q(\varrho) > 0$ for all $\varrho > 0$ sufficiently large because $\lim_{\varrho \rightarrow \infty} q(\varrho) = \infty$. In view of these observations we can see that the investor can realise arbitrarily high risk-free profits by rapidly buying and selling an appropriate amount ϱ of shares, which implies that the market allows for arbitrage and the value function is equal to ∞ . \square

Remark 2. In the model that we have developed, transactions made by the investor have a permanent impact. There are several extensions of the model that can accommodate transient impact. For instance, assuming that $\kappa = \lambda$ for simplicity, we can replace the dynamics given by (2.6) by $X_t = X_t^0 e^{Z_t}$, where

$$Z_t = \lambda \int_{[0,t[} G(t-s) dY_s,$$

for some kernel G . In this context, if we choose $G(t-s) = e^{-\gamma t} e^{\gamma s}$, for some constant $\gamma > 0$, then

$$dZ_t = -\gamma Z_t dt - \lambda d\xi_t^s + \lambda d\xi_t^b.$$

In any case, the resulting optimisation problem's state space involves four variables (namely, t , x , y and z) instead of three (namely, t , x and y). We leave this as well as other extensions accommodating resilience of the stock price for future research. \square

3. Preliminary results. The first result that we establish in this section is concerned with showing that the optimisation problem we have considered is well-posed as well as with proving that simultaneous selling and buying of shares is sub-optimal. To establish the latter claim, we consider any strategy $(\xi^s, \xi^b) \in \mathcal{A}_{\bar{T}, y}$ and we note that there exist unique (\mathcal{F}_t) -adapted increasing càglàd processes $\bar{\xi}^s, \bar{\xi}^b$ such that

$$(3.1) \quad \bar{\xi}_0^s = \bar{\xi}_0^b = 0, \quad -\xi^s + \xi^b = -\bar{\xi}^s + \bar{\xi}^b,$$

and $\bar{\xi}^s + \bar{\xi}^b$ is the total variation process of $-\xi^s + \xi^b$. In particular, $(\bar{\xi}^s, \bar{\xi}^b) \in \mathcal{A}_{\bar{T}, y}$ and there exists a unique (\mathcal{F}_t) -adapted increasing càglàd process ξ such that

$$(3.2) \quad \xi^s = \bar{\xi}^s + \xi \quad \text{and} \quad \xi^b = \bar{\xi}^b + \xi.$$

Lemma 1. *Given any $(T, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}_+$ such that $T \leq \bar{T}$ and any liquidation strategy $(\xi^s, \xi^b) \in \mathcal{A}_{\bar{T}, y}$,*

$$(3.3) \quad \mathbb{E} \left[\sup_{t \in [0, T]} \left(Y_{t+} + X_{t+} + \left| \int_{[0, t]} e^{-\delta s} \left[(X_s - C_s) \circ_\lambda d\xi_s^s - (X_s + C_b) \circ_\kappa d\xi_s^b \right] \right| \right) \right] < \infty.$$

In particular, $J_{T, x, y}(\xi^s, \xi^b)$ is well-defined and real-valued. Furthermore,

$$(3.4) \quad J_{T, x, y}(\xi^s, \xi^b) \leq J_{T, x, y}(\bar{\xi}^s, \bar{\xi}^b) - (C_s + C_b) \mathbb{E} \left[\int_{[0, T]} e^{-\delta t} d\xi_t \right],$$

where $(\bar{\xi}^s, \bar{\xi}^b) \in \mathcal{A}_{\bar{T}, y}$ is the unique liquidation strategy associated with the considerations related to (3.1)–(3.2) and ξ is given by (3.2).

Proof. Throughout the proof we consider any $(T, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}_+$ such that $T \leq \bar{T}$ and any $(\xi^s, \xi^b) \in \mathcal{A}_{\bar{T}, y}$ fixed. In view of the calculations

$$(3.5) \quad de^{-\lambda \xi_t^s} = -\lambda e^{-\lambda \xi_t^s} d(\xi_t^s)^c - e^{-\lambda \xi_t^s} \left[1 - e^{-\lambda \Delta \xi_t^s} \right] = -\lambda e^{-\lambda \xi_t^s} \circ_\lambda d\xi_t^s,$$

$$(3.6) \quad de^{\kappa \xi_t^b} = \kappa e^{\kappa \xi_t^b} d(\xi_t^b)^c + e^{\kappa \xi_t^b} \left[e^{\kappa \Delta \xi_t^b} - 1 \right] = \kappa e^{\kappa \xi_t^b} \circ_\kappa \xi_t^b,$$

which follow from Itô's formula, and (2.4)–(2.6), we can see that

$$\begin{aligned} & \int_{[0, T]} e^{-\delta t} \left[(X_t - C_s) \circ_\lambda d\xi_t^s - (X_t + C_b) \circ_\kappa d\xi_t^b \right] \\ &= \int_{[0, T]} e^{-\delta t} \left[-\frac{1}{\lambda} X_t^0 e^{\kappa \xi_t^b} de^{-\lambda \xi_t^s} - \frac{1}{\kappa} X_t^0 e^{-\lambda \xi_t^s} de^{\kappa \xi_t^b} - C_s d\xi_t^s - C_b d\xi_t^b \right]. \end{aligned}$$

Combining this expression with the identities

$$\begin{aligned} X_T^0 e^{-\lambda \xi_{T+}^s} - x &= \int_{[0, T]} X_t^0 de^{-\lambda \xi_t^s} + \mu \int_0^T X_t^0 e^{-\lambda \xi_t^s} dt + M_T^\lambda, \\ X_T^0 e^{\kappa \xi_{T+}^b} - x &= \int_{[0, T]} X_t^0 de^{\kappa \xi_t^b} + \mu \int_0^T X_t^0 e^{\kappa \xi_t^b} dt + M_T^\kappa, \end{aligned}$$

where

$$M_T^\lambda = \sigma \int_0^T X_t^0 e^{-\lambda \xi_t^s} dW_t \quad \text{and} \quad M_T^\kappa = \sigma \int_0^T X_t^0 e^{\kappa \xi_t^b} dW_t,$$

and the fact that ξ^s, ξ^b are positive increasing processes satisfying (2.10), we obtain

$$\begin{aligned} & \sup_{t \in [0, T]} \left| \int_{[0, t]} e^{-\delta s} \left[(X_s - C_s) \circ_\lambda d\xi_s^s - (X_s + C_b) \circ_\kappa d\xi_s^b \right] \right| \\ & \leq -\frac{1}{\lambda} e^{\kappa \xi_{T+}^b} \int_{[0, T]} X_t^0 de^{-\lambda \xi_t^s} + \frac{1}{\kappa} \int_{[0, T]} X_t^0 de^{\kappa \xi_t^b} + C_s \xi_{T+}^s + C_b \xi_{T+}^b \\ & \leq \frac{x}{\lambda} + \frac{1}{\kappa} X_T^0 e^{\kappa \xi_{T+}^b} + \left(\frac{|\mu|}{\lambda} + \frac{|\mu|}{\kappa} \right) e^{\kappa \xi_{T+}^b} \int_0^T X_t^0 dt + \frac{1}{\lambda} M_T^\lambda - \frac{1}{\kappa} M_T^\kappa \\ (3.7) \quad & + C_s y + (C_s + C_b) \xi_{T+}^b. \end{aligned}$$

The stochastic integrals M^λ , M^κ are square-integrable martingales because Itô's isometry, Hölder's inequality and (2.11) in Assumption 1 imply that

$$\begin{aligned}\mathbb{E} \left[\left(M_T^\lambda \right)^2 \right] &= \sigma^2 \int_0^T \mathbb{E} \left[\left(X_t^0 e^{-\lambda \xi_t^s} \right)^2 \right] dt \leq \sigma^2 \int_0^T \mathbb{E} \left[\left(X_t^0 \right)^2 \right] dt < \infty \\ \mathbb{E} \left[\left(M_T^\kappa \right)^2 \right] &= \sigma^2 \int_0^T \mathbb{E} \left[\left(X_t^0 e^{\kappa \xi_t^b} \right)^2 \right] dt \leq \sigma^2 \sqrt{\mathbb{E} \left[e^{4\kappa \xi_{T+}^b} \right]} \int_0^T \sqrt{\mathbb{E} \left[\left(X_t^0 \right)^4 \right]} dt < \infty.\end{aligned}$$

Also, Jensen's inequality and (2.10)–(2.11) imply that

$$\mathbb{E} \left[\sup_{t \in [0, T]} Y_{t+} \right] \leq y + \mathbb{E} \left[\xi_{T+}^b \right] \leq y + \frac{1}{4\kappa} \ln \mathbb{E} \left[e^{4\kappa \xi_{T+}^b} \right] < \infty,$$

while, Hölder's inequality and (2.6), which X satisfies, imply that

$$\mathbb{E} \left[\sup_{t \in [0, T]} X_{t+} \right] \leq \mathbb{E} \left[e^{\kappa \xi_{T+}^b} \sup_{t \in [0, T]} X_t^0 \right] \leq \left(\mathbb{E} \left[e^{4\kappa \xi_{T+}^b} \right] \right)^{\frac{1}{4}} \left(\mathbb{E} \left[\sup_{t \in [0, T]} \left(X_t^0 \right)^{\frac{4}{3}} \right] \right)^{\frac{3}{4}} < \infty.$$

In view of these considerations, we can see that the right-hand side of (3.7) is a positive random variable with finite expectation and that (3.3) holds true.

To prove (3.4), we note that

$$J_{T,x,y}(\xi^s, \xi^b) = J_{T,x,y}(\bar{\xi}^s, \bar{\xi}^b) + \mathbb{E} \left[\sum_{t \in [0, T]} e^{-\delta t} X_t Z_t \right] - (C_s + C_b) \mathbb{E} \left[\int_{[0, T]} e^{-\delta t} d\xi_t \right],$$

where

$$Z_t = \frac{1}{\lambda} \left[1 - e^{-\lambda(\Delta \bar{\xi}_t^s + \Delta \xi_t)} \right] - \frac{1}{\lambda} \left[1 - e^{-\lambda \Delta \bar{\xi}_t^s} \right] - \frac{1}{\kappa} \left[e^{\kappa(\Delta \bar{\xi}_t^b + \Delta \xi_t)} - 1 \right] + \frac{1}{\kappa} \left[e^{\kappa \Delta \bar{\xi}_t^b} - 1 \right].$$

This expression implies the required result because

$$q(z) := \frac{1}{\lambda} \left[1 - e^{-\lambda(a_1+z)} \right] - \frac{1}{\lambda} \left[1 - e^{-\lambda a_1} \right] - \frac{1}{\kappa} \left[e^{\kappa(a_2+z)} - 1 \right] + \frac{1}{\kappa} \left[e^{\kappa a_2} - 1 \right] < 0$$

for all constants $a_1, a_2 \geq 0$ and $z > 0$, the inequality following from the calculations $q(0) = 0$ and $q'(z) = e^{-\lambda(a_1+z)} - e^{\kappa(a_2+z)} < 0$.

The next result is concerned with a preliminary characterisation of the optimal strategy as well as with the solution to a special case of the optimisation problem.

Lemma 2. *Given a time horizon $\bar{T} \in (0, \infty]$ and any $(x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+$, the following statements are true:*

(I) *If $\delta \geq \mu$, then buying is never part of an optimal strategy and the value function satisfies $v(\bar{T}, x, y) = \sup_{\xi^s \in \mathcal{A}_{\bar{T}, y}^s} I_{\bar{T}, x, y}(\xi^s, 0)$.*

(II) *If $\delta \geq \mu$ and $C_s = 0$, then it is optimal to sell all available shares at time 0 and the value function is given by $v(\bar{T}, x, y) = \frac{1}{\lambda} x \left[1 - e^{-\lambda y} \right]$.*

(III) If $\bar{T} < \infty$, $\delta < \mu$ and buying is not allowed, then it is optimal to sell all available shares at time \bar{T} , namely,

$$(3.8) \quad \sup_{\xi^s \in \mathcal{A}_{\bar{T},y}^s} I_{\bar{T},x,y}(\xi^s, 0) = \sup_{\xi^s \in \mathcal{A}_{\bar{T},y}^s} J_{\bar{T},x,y}(\xi^s, 0) = \frac{1}{\lambda} x e^{(\mu-\delta)\bar{T}} [1 - e^{-\lambda y}] - e^{-\delta\bar{T}} C_s y.$$

Proof. To prove (I), we consider any $(\xi^s, \xi^b) \in \mathcal{A}_{\bar{T},y}$ and any $T \in \mathbb{R}_+$. Using the integration by parts formula, we can check that (2.3) implies that

$$\begin{aligned} d\left(e^{-\delta t} X_t\right) &= d\left(e^{-\delta t} X_t^0 e^{-\lambda \xi_t^s + \kappa \xi_t^b}\right) \\ &= (\mu - \delta) e^{-\delta t} X_t^0 e^{-\lambda \xi_t^s + \kappa \xi_t^b} dt + e^{-\delta t} X_t^0 e^{\kappa \xi_t^b} de^{-\lambda \xi_t^s} \\ &\quad + e^{-\delta t} X_t^0 e^{-\lambda \xi_t^s} de^{\kappa \xi_t^b} + \sigma e^{-\delta t} X_t^0 e^{-\lambda \xi_t^s + \kappa \xi_t^b} dW_t. \end{aligned}$$

In view of these identities and (3.5)–(3.6), we can see that the functional defined by (2.8) admits the expressions

$$\begin{aligned} J_{T,x,y}(\xi^s, \xi^b) &= \mathbb{E} \left[\int_{[0,T]} e^{-\delta t} \left[-\frac{1}{\lambda} X_t^0 e^{\kappa \xi_t^b} de^{-\lambda \xi_t^s} - \frac{1}{\kappa} X_t^0 e^{-\lambda \xi_t^s} de^{\kappa \xi_t^b} - C_s d\xi_t^s - C_b d\xi_t^b \right] \right] \\ &= \mathbb{E} \left[-\frac{\lambda - \kappa}{\kappa \lambda} \int_{[0,T]} e^{-\delta t} X_t^0 e^{-\lambda \xi_t^s} de^{\kappa \xi_t^b} - \frac{\delta - \mu}{\lambda} \int_0^T e^{-\delta t} X_t^0 e^{-\lambda \xi_t^s + \kappa \xi_t^b} dt \right. \\ (3.9) \quad &\left. + \frac{x}{\lambda} - \frac{1}{\lambda} e^{-\delta T} X_T^0 e^{-\lambda \xi_{T+}^s + \kappa \xi_{T+}^b} - \int_{[0,T]} e^{-\delta t} [C_s d\xi_t^s + C_b d\xi_t^b] \right]. \end{aligned}$$

If we define

$$\tilde{\xi}_t^s = \sup_{0 \leq u \leq t} \left(\xi_u^s - \xi_u^b \right)^+ \quad \text{and} \quad \hat{\xi}_t^s = \tilde{\xi}_t^s + \xi_t^b,$$

then we can check that

$$\tilde{\xi}_t^s \leq \xi_t^s \leq \xi_t^s + \left(\xi_t^b - \xi_t^s \right)^+ = \left(\xi_t^s - \xi_t^b \right)^+ + \xi_t^b \leq \hat{\xi}_t^s.$$

In view of these inequalities, the calculation

$$\int_{[0,T]} e^{-\delta t} d\left(\xi_t^s - \tilde{\xi}_t^s\right) = e^{-\delta T} \left(\xi_{T+}^s - \tilde{\xi}_{T+}^s \right) + \delta \int_0^T e^{-\delta t} \left(\xi_t^s - \tilde{\xi}_t^s \right) dt \geq 0$$

and (3.4), we can see that, if $\delta \geq \mu$, then (3.9) implies that

$$\begin{aligned} J_{T,x,y}(\xi^s, \xi^b) &\leq J_{T,x,y}(\hat{\xi}^s, \xi^b) + C_s \mathbb{E} \left[\int_{[0,T]} e^{-\delta t} d\left(\hat{\xi}_t^s - \xi_t^s\right) \right] \\ &\leq J_{T,x,y}(\tilde{\xi}^s, 0) - (C_s + C_b) \mathbb{E} \left[\int_{[0,T]} e^{-\delta t} d\xi_t^b \right] + C_s \mathbb{E} \left[\int_{[0,T]} e^{-\delta t} d\left(\hat{\xi}_t^s - \xi_t^s\right) \right] \\ &\leq J_{T,x,y}(\tilde{\xi}^s, 0) - C_s \mathbb{E} \left[\int_{[0,T]} e^{-\delta t} d\left(\xi_t^s - \tilde{\xi}_t^s\right) \right] \\ &\leq J_{T,x,y}(\tilde{\xi}^s, 0). \end{aligned}$$

Part (I) now follows from (2.7) and the definition (2.9) of the value function v .

To establish (II) and (III), we now assume that $\bar{T} < \infty$ and we consider any $\xi^s \in \mathcal{A}_{\bar{T},y}^s$. The admissibility of the strategy $(\xi^s, 0)$ implies that $\xi_{\bar{T}+}^s = y$ (see (2.1)). Therefore, (3.9) implies that

$$J_{\bar{T},x,y}(\xi^s, 0) = \mathbb{E} \left[\frac{x}{\lambda} - \frac{1}{\lambda} e^{-\delta \bar{T}} X_{\bar{T}}^0 e^{-\lambda y} - \frac{\delta - \mu}{\lambda} \int_0^{\bar{T}} e^{-\delta t} X_t^0 e^{-\lambda \xi_t^s} dt - C_s \int_{[0,\bar{T}]} e^{-\delta t} d\xi_t^s \right].$$

If $\delta \geq \mu$ and $C_s = 0$, then this expression is plainly maximised by the choice $\xi_t^s = y$ for all $t > 0$, and (II) follows. On the other hand, if $\delta < \mu$, then this expression is maximised by $\xi_t^s = y \mathbf{1}_{(\bar{T}, \infty)}(t)$ for all $t > 0$. Therefore, if $\delta < \mu$ and buying is not allowed, then selling all shares at time \bar{T} is optimal. Since this strategy has payoff

$$\mathbb{E} \left[\frac{1}{\lambda} X_{\bar{T}}^0 [1 - e^{-\lambda y}] - e^{-\delta \bar{T}} C_s y \right] = \frac{1}{\lambda} x e^{(\mu - \delta) \bar{T}} [1 - e^{-\lambda y}] - e^{-\delta \bar{T}} C_s y,$$

(III) follows.

4. The finite time horizon case ($\bar{T} < \infty$). We expect that the value function v of the stochastic control problem formulated in §2 identifies with an appropriate solution $w : \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}_+ \rightarrow \mathbb{R}$ to the HJB equation

$$(4.1) \quad \max \left\{ -w_t(t, x, y) + \mathcal{L}w(t, x, y), -\lambda x w_x(t, x, y) - w_y(t, x, y) + x - C_s, \right. \\ \left. \kappa x w_x(t, x, y) + w_y(t, x, y) - x - C_b \right\} = 0,$$

with boundary condition

$$(4.2) \quad w(0, x, y) = \frac{1}{\lambda} x [1 - e^{-\lambda y}] - C_s y,$$

where

$$(4.3) \quad \mathcal{L}w(t, x, y) = \frac{1}{2} \sigma^2 x^2 w_{xx}(t, x, y) + \mu x w_x(t, x, y) - \delta w(t, x, y).$$

To obtain qualitative understanding of this equation, we consider the following heuristic arguments. Suppose that, at a given time, the investor's horizon is $t > 0$, the share price is $x > 0$ and the investor holds an amount $y > 0$ of shares. At that time, the investor is faced with three possible actions. The first one is to wait for a short time Δt and then continue optimally. Bellman's principle of optimality implies that this possibility, which is not necessarily optimal, is associated with the inequality

$$v(t, x, y) \geq \mathbb{E} \left[e^{-\delta \Delta t} v(t - \Delta t, X_{\Delta t}, y) \right].$$

Applying Itô's formula and dividing by Δt before letting $\Delta t \downarrow 0$, we obtain

$$(4.4) \quad -v_t(t, x, y) + \frac{1}{2} \sigma^2 x^2 v_{xx}(t, x, y) + \mu x v_x(t, x, y) - \delta v(t, x, y) \leq 0.$$

The second possibility is to sell a small amount $\varepsilon > 0$ of shares, and then continue optimally. This action is associated with the inequality

$$v(t, x, y) \geq v(t, x - \lambda x \varepsilon, y - \varepsilon) + (x - C_s) \varepsilon.$$

Rearranging terms and letting $\varepsilon \downarrow 0$, we obtain

$$(4.5) \quad -\lambda x v_x(t, x, y) - v_y(t, x, y) + x - C_s \leq 0.$$

The third possible action is to buy a small amount $\varepsilon > 0$ of shares, and then continue optimally, which is associated with the inequality

$$v(t, x, y) \geq v(t, x + \kappa x \varepsilon, y + \varepsilon) - (x + C_b) \varepsilon.$$

Passing to the limit $\varepsilon \downarrow 0$, we can see that

$$(4.6) \quad \kappa x v_x(t, x, y) + v_y(t, x, y) - x - C_b \leq 0.$$

The Markovian character of the problem implies that one of these three possibilities should be optimal and one of (4.4)–(4.6) should hold with equality at any point in the state space. It follows that the problem's value function v should identify with an appropriate solution w of the HJB equation (5.2). Also, the boundary condition in (4.2) follows from the requirement that the investor must liquidate all share holdings at the end of the planning horizon (see also (2.1) and (2.4)).

We now prove a verification theorem that associates a smooth solution to the HJB equation (4.1)–(4.2) with the control problem's value function and can be used to identify an optimal liquidation strategy. To this end, we consider the sets

$$\begin{aligned} \mathcal{W} &= \{(t, x, y) \in [0, T] \times \mathbb{R}_+^* \times \mathbb{R}_+ \mid -w_t(t, x, y) + \mathcal{L}w(t, x, y) = 0\}, \\ \mathcal{S} &= \{(t, x, y) \in [0, T] \times \mathbb{R}_+^* \times \mathbb{R}_+ \mid \lambda x w_x(t, x, y) + w_y(t, x, y) - x + C_s = 0\}, \\ \mathcal{B} &= \{(t, x, y) \in [0, T] \times \mathbb{R}_+^* \times \mathbb{R}_+ \mid \kappa x w_x(t, x, y) + w_y(t, x, y) - x - C_b = 0\}, \end{aligned}$$

and we call them the “waiting” region, the “selling” region and the “buying” region, respectively, consistently with the heuristics that we have discussed above. It is worth noting that the liquidation strategy that sells all available shares at time 0 and takes no further action gives rise to the inequalities

$$v(T, x, y) \geq \frac{1}{\lambda} x [1 - e^{-\lambda y}] - C_s y \geq -C_s y,$$

which suggest (4.10) and the lower bound in (4.12). Also, we are going to prove in Proposition 4 below that the value function v does admit an upper bound such as the one in (4.12).

Proposition 3. *Given a time horizon $\bar{T} \in (0, \infty)$ and an initial condition $(x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+$, the following statements are true:*

(I) *If a function $w : [0, \bar{T}] \times \mathbb{R}_+^* \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is $C^{1,2,1}$ and satisfies the inequalities*

$$(4.7) \quad -w_t(t, x, y) + \mathcal{L}w(t, x, y) \leq 0,$$

$$(4.8) \quad -\lambda x w_x(t, x, y) - w_y(t, x, y) + x - C_s \leq 0,$$

$$(4.9) \quad \kappa x w_x(t, x, y) + w_y(t, x, y) - x - C_b \leq 0,$$

$$(4.10) \quad -C_s y - w(t, x, y) \leq 0,$$

for all $(t, x, y) \in [0, \bar{T}] \times \mathbb{R}_+^* \times \mathbb{R}_+$, then

$$(4.11) \quad v(\bar{T}, x, y) \leq w(\bar{T}, x, y) \quad \text{for all } (x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+.$$

(II) If a function $w : [0, \bar{T}] \times \mathbb{R}_+^* \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is a $C^{1,2,1}$ solution to the HJB equation (4.1)–(4.2) such that

$$(4.12) \quad -C_s y \leq w(t, x, y) \leq Kx,$$

for all $(t, x, y) \in [0, \bar{T}] \times \mathbb{R}_+^* \times \mathbb{R}_+$, for some constant $K > 0$, and there exists an admissible strategy $(\xi^{s*}, \xi^{b*}) \in \mathcal{A}_{\bar{T}, y}$ such that

$$(4.13) \quad (X_t^*, Y_t^*) \in \mathcal{W} \quad \text{for all } t \geq 0, \mathbb{P}\text{-a.s.},$$

$$(4.14) \quad \xi_{t+}^{s*} = \int_{[0, t]} \mathbf{1}_{\{(X_t^*, Y_t^*) \in \mathcal{S}\}} d\xi_t^{s*} \quad \text{and} \quad \xi_{t+}^{b*} = \int_{[0, t]} \mathbf{1}_{\{(X_t^*, Y_t^*) \in \mathcal{B}\}} d\xi_t^{b*} \quad \text{for all } t \geq 0, \mathbb{P}\text{-a.s.},$$

where X^* and Y^* are the associated share price and shares held processes, then

$$(4.15) \quad v(\bar{T}, x, y) = w(\bar{T}, x, y) \quad \text{for all } (x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+,$$

and (ξ^{s*}, ξ^{b*}) is an optimal liquidation strategy.

Proof. Fix any initial condition $(x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+$ and any admissible strategy $(\xi^s, \xi^b) \in \mathcal{A}_{\bar{T}, y}$. In view of Itô-Tanaka-Meyer's formula and the left-continuity of the processes X, Y , we can see that

$$\begin{aligned} e^{-\delta t} w(\bar{T} - t, X_{t+}, Y_{t+}) &= w(\bar{T}, x, y) + \int_0^t e^{-\delta s} [-w_t(s, X_s, Y_s) + \mathcal{L}w(s, X_s, Y_t)] ds \\ &\quad + M_t - \int_0^t e^{-\delta s} [\lambda X_s w_x(s, X_s, Y_s) + w_y(s, X_s, Y_s)] d(\xi^s)_s^c \\ &\quad + \int_0^t e^{-\delta s} [\kappa X_s w_x(s, X_s, Y_s) + w_y(s, X_s, Y_s)] d(\xi^b)_s^c \\ &\quad + \sum_{0 \leq s \leq t} e^{-\delta s} [w(s, X_{s+}, Y_{s+}) - w(s, X_s, Y_s)], \end{aligned}$$

where

$$(4.16) \quad M_t = \sigma \int_0^t e^{-\delta s} X_s w_x(s, X_s, Y_s) dW_s.$$

Combining this calculation with the observation that

$$\begin{aligned} &w(s, X_{s+}, Y_{s+}) - w(s, X_s, Y_s) \\ &= \int_0^{\Delta \xi_s^s} \frac{dw(s, X_s e^{-\lambda u}, Y_s - u)}{du} du + \int_0^{\Delta \xi_s^b} \frac{dw(s, X_s e^{\kappa u}, Y_s + u)}{du} du \\ &= - \int_0^{\Delta \xi_s^s} \left[\lambda X_s e^{-\lambda u} w_x(s, X_s e^{-\lambda u}, Y_s - u) + w_y(s, X_s e^{-\lambda u}, Y_s - u) \right] du \\ &\quad + \int_0^{\Delta \xi_s^b} \left[\kappa X_s e^{\kappa u} w_x(s, X_s e^{\kappa u}, Y_s + u) + w_y(s, X_s e^{\kappa u}, Y_s + u) \right] du \end{aligned}$$

and (2.4)–(2.5), we obtain

$$\begin{aligned}
& \int_{[0,t]} e^{-\delta s} \left[(X_s - C_s) \circ_\lambda d\xi_s^s - (X_s + C_b) \circ_\kappa d\xi_s^b \right] + e^{-\delta t} w(\bar{T} - t, X_{t+}, Y_{t+}) \\
&= w(\bar{T}, x, y) + \int_0^t e^{-\delta s} [-w_t(s, X_s, Y_s) + \mathcal{L}w(s, X_s, Y_s)] ds + M_t \\
&\quad + \int_0^t e^{-\delta s} [-\lambda X_s w_x(s, X_s, Y_s) - w_y(s, X_s, Y_s) + X_s - C_s] d(\xi^s)_s^c \\
&\quad + \int_0^t e^{-\delta s} [\kappa X_s w_x(s, X_s, Y_s) + w_y(s, X_s, Y_s) - X_s - C_b] d(\xi^b)_s^c \\
&\quad + \sum_{0 \leq s \leq t} e^{-\delta s} \int_0^{\Delta \xi_s^s} \left[-\lambda X_s e^{-\lambda u} w_x(s, X_s e^{-\lambda u}, Y_s - u) \right. \\
&\quad \quad \quad \left. - w_y(s, X_s e^{-\lambda u}, Y_s - u) + X_s e^{-\lambda u} - C_s \right] du \\
&\quad + \sum_{0 \leq s \leq t} e^{-\delta s} \int_0^{\Delta \xi_s^b} \left[\kappa X_s e^{\kappa u} w_x(s, X_s e^{\kappa u}, Y_s + u) \right. \\
&\quad \quad \quad \left. + w_y(s, X_s e^{\kappa u}, Y_s + u) - X_s e^{\kappa u} - C_b \right] du.
\end{aligned} \tag{4.17}$$

Since w satisfies (4.7)–(4.10), this calculation implies that

$$\int_{[0,t]} e^{-\delta s} \left[(X_s - C_s) \circ_\lambda d\xi_s^s - (X_s + C_b) \circ_\kappa d\xi_s^b \right] - C_s e^{-\delta t} Y_{t+} \leq w(\bar{T}, x, y) + M_t. \tag{4.18}$$

In particular, we can see that

$$\int_{[0,\bar{T}]} e^{-\delta s} \left[(X_s - C_s) \circ_\lambda d\xi_s^s - (X_s + C_b) \circ_\kappa d\xi_s^b \right] \leq w(\bar{T}, x, y) + M_{\bar{T}} \tag{4.19}$$

thanks to the fact that $Y_{\bar{T}+} = 0$. In view of (3.3) and (4.18), we can see that $\inf_{0 \leq t \leq \bar{T}} M_t$ is bounded from below by an integrable random variable. Therefore, the stochastic integral M is a supermartingale. In light of this observation, we can take expectations in (4.19) to obtain

$$J_{\bar{T},x,y}(\xi^s, \xi^b) = \mathbb{E} \left[\int_{[0,\bar{T}]} e^{-\delta s} \left[(X_s - C_s) \circ_\lambda d\xi_s^s - (X_s + C_b) \circ_\kappa d\xi_s^b \right] \right] \leq w(\bar{T}, x, y).$$

It follows that (4.11) is true because (ξ^s, ξ^b) has been an arbitrary admissible strategy, and part (I) of the proposition has been established.

If a strategy $(\xi^{s*}, \xi^{b*}) \in \mathcal{A}_{\bar{T},y}$ satisfies (4.13)–(4.14), then we can check that (4.17) yields

$$\begin{aligned}
& \int_{[0,t]} e^{-\delta s} \left[(X_s^* - C_s) \circ_\lambda d\xi_s^{s*} - (X_s^* + C_b) \circ_\kappa d\xi_s^{b*} \right] + K e^{-\delta t} X_{t+}^* \geq w(\bar{T}, x, y) + M_t^*, \\
& \int_{[0,\bar{T}]} e^{-\delta s} \left[(X_s^* - C_s) \circ_\lambda d\xi_s^{s*} - (X_s^* + C_b) \circ_\kappa d\xi_s^{b*} \right] = w(\bar{T}, x, y) + M_{\bar{T}}^*,
\end{aligned}$$

instead of (4.18), (4.19). The inequality and (3.3) imply that $\sup_{0 \leq t \leq \bar{T}} M_t$ is bounded from above by an integrable random variable. Therefore, M is a submartingale and we can take expectations to the identity to obtain

$$J_{\bar{T},x,y}(\xi^{s*}, \xi^{b*}) = \mathbb{E} \left[\int_{[0, \bar{T}]} e^{-\delta s} \left[(X_s^* - C_s) \circ_{\lambda} d\xi_s^{s*} - (X_s^* + C_b) \circ_{\kappa} d\xi_s^{b*} \right] \right] \geq w(\bar{T}, x, y),$$

which, combined with part (I) of the proposition, implies (4.15) as well as the optimality of (ξ^{s*}, ξ^{b*}) .

We now use Lemma 2 and the verification result that we have just established to derive the following characterisations of the solution to the control problem. It is worth noting that, if $\delta < \mu$, then the solution to the control problem does not depend continuously on the time horizon \bar{T} . Indeed, if we compare (III) with (IV), then we can see that

$$(4.20) \quad v(\bar{T}, x, y) \leq \frac{1}{\lambda} x e^{(\mu-\delta)\bar{T}_*} \quad \text{for all } \bar{T} \in [0, \bar{T}_*],$$

while, $v(\bar{T}, x, y) = \infty$ for all $\bar{T} > \bar{T}_*$, where $\bar{T}_* = \frac{1}{\mu-\delta} \ln \frac{\lambda}{\kappa}$.

Proposition 4. *Given a time horizon $\bar{T} \in (0, \infty)$ and an initial condition $(x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+$, the following statements are true:*

(I) *If $\delta \geq \mu$, then buying is never part of the optimal tactics. In this case, the value function satisfies $v(\bar{T}, x, y) = \sup_{\xi^s \in \mathcal{A}_{\bar{T}, y}^s} I_{\bar{T}, x, y}(\xi^s, 0)$ as well as*

$$(4.21) \quad \frac{1}{\lambda} x \left[1 - e^{-\lambda y} \right] - C_s y \leq v(\bar{T}, x, y) \leq \frac{1}{\lambda} x \left[1 - e^{-\lambda y} \right].$$

(II) *If $\delta \geq \mu$ and $C_s = 0$, then it is optimal to sell all available shares at time 0 and the value function is given by*

$$(4.22) \quad v(\bar{T}, x, y) = \frac{1}{\lambda} x \left[1 - e^{-\lambda y} \right].$$

(III) *If $\delta < \mu$ and $\kappa e^{(\mu-\delta)\bar{T}} \leq \lambda$, then the value function satisfies*

$$(4.23) \quad \frac{1}{\lambda} x \left[1 - e^{-\lambda y} \right] - C_s y \leq v(\bar{T}, x, y) \leq \frac{1}{\lambda} x e^{(\mu-\delta)\bar{T}}.$$

In this case, the optimal strategy does involve buying of shares.

(IV) *If $\delta < \mu$ and $\kappa e^{(\mu-\delta)\bar{T}} > \lambda$, then $v(\bar{T}, x, y) = \infty$. In this case, arbitrarily high payoffs can be achieved only by strategies that do involve buying of shares.*

Proof. The first claim in (I) is a restatement of Lemma 2.(I), while the lower bound in (4.21) is the payoff of the strategy that liquidates all available shares at time 0. If we define $w(t, x, y) = \frac{1}{\lambda} x \left[1 - e^{-\lambda y} \right]$, then we can check that w satisfies all of the inequalities (4.7)–(4.10), and the upper bound in (4.21) follows from Proposition 3.(I). Also, part (II) is a restatement of Lemma 2.(II).

To prove (III), we consider the function w defined by $w(t, x, y) = \Gamma x e^{(\mu-\delta)t}$, for some constant $\Gamma > 0$. This function satisfies (4.7)–(4.10) if and only if

$$\frac{x - C_s}{\lambda} \leq \Gamma x e^{(\mu-\delta)t} \leq \frac{x + C_b}{\kappa}.$$

If $\mu > \delta$, then these inequalities can be true for all $t \in [0, \bar{T}]$ and $x > 0$ if and only if

$$\frac{x - C_s}{\lambda x} \leq \Gamma \leq \frac{x + C_b}{\kappa x} e^{-(\mu - \delta)\bar{T}} \text{ for all } x > 0 \quad \Leftrightarrow \quad \frac{1}{\lambda} \leq \Gamma \leq \frac{1}{\kappa} e^{-(\mu - \delta)\bar{T}}.$$

The last set of these inequalities can be true if and only if $\kappa e^{(\mu - \delta)\bar{T}} \leq \lambda$, in which case, we can see that the value $\Gamma = \frac{1}{\lambda}$ is an appropriate one by considering the very first of these inequalities, and the upper bound in (4.23) follows from Proposition 3.(I).

In the context of (III), if buying were not part of the optimal tactics, then the value function would be given by (3.8) in Lemma 2. However, this possibility contradicts Proposition 3 because this function fails to satisfy the HJB equation (4.1).

Finally, we consider the strategy that buys $z > 0$ shares at time 0 and sells all shares at time \bar{T} . The payoff that this strategy yields is equal to

$$\begin{aligned} & -\frac{1}{\kappa} x [e^{\kappa z} - 1] + \frac{1}{\lambda} \mathbb{E} \left[e^{-\delta \bar{T}} X_{\bar{T}}^0 e^{\kappa z} \right] \left[1 - e^{-\lambda(z+y)} \right] - C_b z - e^{-\delta \bar{T}} C_s(z+y) \\ & = x e^{\kappa z} \left(\frac{1}{\lambda} e^{(\mu - \delta)\bar{T}} \left[1 - e^{-\lambda(z+y)} \right] - \frac{1}{\kappa} \right) + \frac{x}{\kappa} - C_b z - e^{-\delta \bar{T}} C_s(z+y). \end{aligned}$$

If $\kappa e^{(\mu - \delta)\bar{T}} > \lambda$, then this payoff converges to ∞ as $z \rightarrow \infty$, which establishes the identity $v(\bar{T}, x, y) = \infty$. In particular, arbitrarily high payoffs can be achieved only by strategies involving buying of shares thanks to the bound (3.8) in Lemma 2.

Remark 3. A round-trip trade over a time horizon \bar{T} is any trading strategy that involves 0 net buying or selling of shares by time \bar{T} , i.e., any trading strategy (ζ^s, ζ^b) such that $\zeta_{\bar{T}+}^s = \zeta_{\bar{T}+}^b$. We say that a round-trip trade is admissible if the maximum number of stock shares that it can be short is bounded by a constant, namely, if

$$(4.24) \quad \zeta_{\bar{T}+}^s - \zeta_{\bar{T}+}^b \leq y,$$

for some constant $y > 0$, which may depend on the round-trip trade itself. (It is worth noting that this condition of admissibility is weaker than the one in Gatheral, Schied and Slynko [19] because it does not impose a bound on the maximum number of shares that the strategy can be long.) A price manipulation is a round-trip trade with strictly positive execution payoff, namely, a trading strategy (ζ^s, ζ^b) such that

$$\zeta_{\bar{T}+}^s = \zeta_{\bar{T}+}^b \quad \text{and} \quad J_{\bar{T}, x, y}(\zeta^s, \zeta^b) > 0,$$

for some $\bar{T} > 0$.

The assumption $\kappa \leq \lambda$ and the inequality $\delta \geq \mu$ are sufficient conditions for the model to allow for no price manipulation. To see this claim, it suffices to show that

$$(4.25) \quad \mathbb{E} \left[\int_{[0, \bar{T}]} e^{-\delta t} X_t \circ_{\lambda} d\zeta_t^s - X_t \circ_{\kappa} d\zeta_t^b \right] \leq 0$$

for every round-trip trade (ζ^s, ζ^b) because

$$J_{\bar{T}, x, y}(\zeta^s, \zeta^b) \leq \mathbb{E} \left[\int_{[0, \bar{T}]} e^{-\delta t} X_t \circ_{\lambda} d\zeta_t^s - X_t \circ_{\kappa} d\zeta_t^b \right].$$

To this end, we consider any round-trip trade (ζ^s, ζ^b) as above and we define the liquidation strategy $(\xi^s, \xi^b) \in \mathcal{A}_{\bar{T}, y}$ by

$$(4.26) \quad \xi_t^s = \begin{cases} \zeta_t^s, & \text{if } t \leq \bar{T}, \\ y, & \text{if } t > \bar{T}, \end{cases} \quad \text{and} \quad \xi_t^b = \begin{cases} \zeta_t^b, & \text{if } t \leq \bar{T}, \\ \zeta_{\bar{T}+}^b, & \text{if } t > \bar{T}, \end{cases}$$

where $y > 0$ is any bound as in (4.24). This strategy puts us in the context of an investor who starts with y shares and follows the round-trip trade up to time \bar{T} when she sells all available shares. (Note that this scenario may involve simultaneous buying and selling at time \bar{T} .) We then calculate

$$(4.27) \quad \begin{aligned} & \mathbb{E} \left[\int_{[0, \bar{T}]} e^{-\delta t} X_t \circ_{\lambda} d\zeta_t^s - X_t \circ_{\kappa} d\zeta_t^b \right] \\ &= \mathbb{E} \left[\int_{[0, T]} e^{-\delta t} X_t \circ_{\lambda} d\zeta_t^s - X_t \circ_{\kappa} d\zeta_t^b \right] - \frac{1}{\lambda} \mathbb{E} \left[e^{-\delta \bar{T}} X_{\bar{T}} \right] \left[1 - e^{-\lambda y} \right] \\ &\leq v(\bar{T}, x, y) - \frac{1}{\lambda} \mathbb{E} \left[e^{-\delta \bar{T}} X_{\bar{T}} \right] \left[1 - e^{-\lambda y} \right], \end{aligned}$$

where v is the value function of the optimal execution problem for $C_s = C_b = 0$. In view of the identity $\zeta_{\bar{T}+}^s = \zeta_{\bar{T}+}^b$ characterising the round-trip trade, the assumption $\kappa \leq \lambda$, (2.6) and (4.26), we can see that

$$(4.28) \quad \mathbb{E} \left[e^{-\delta \bar{T}} X_{\bar{T}} \right] = \mathbb{E} \left[e^{-\delta \bar{T}} X_{\bar{T}}^0 e^{-\lambda \zeta_{\bar{T}}^s + \kappa \zeta_{\bar{T}}^b} \right] \leq \mathbb{E} \left[e^{-\delta \bar{T}} X_{\bar{T}}^0 \right] = x e^{(\mu - \delta) \bar{T}}.$$

Combining (4.27) with (4.28) and Proposition 4.(II), we can see that

$$\mathbb{E} \left[\int_{[0, \bar{T}]} e^{-\delta t} X_t \circ_{\lambda} d\zeta_t^s - X_t \circ_{\kappa} d\zeta_t^b \right] \leq \frac{1}{\lambda} x \left[1 - e^{-\lambda y} \right] - \frac{1}{\lambda} x e^{(\mu - \delta) \bar{T}} \left[1 - e^{-\lambda y} \right] \leq 0.$$

It follows that there can be no price manipulation over any time horizon \bar{T} if $\kappa \leq \lambda$ and $\delta \geq \mu$.

On the other hand, the assumption $\kappa \leq \lambda$ and the inequality $\delta < \mu$ allow for market manipulation provided the time horizon \bar{T} is sufficiently long. Indeed, the round-trip trade that buys $z > 0$ shares at time 0 and sells all shares at time \bar{T} has payoff

$$\begin{aligned} & -\frac{1}{\kappa} x \left[e^{\kappa z} - 1 \right] + \frac{1}{\lambda} \mathbb{E} \left[e^{-\delta \bar{T}} X_{\bar{T}}^0 e^{\kappa z} \right] \left[1 - e^{-\lambda z} \right] - C_b z - e^{-\delta \bar{T}} C_s z \\ &= x e^{\kappa z} \left(\frac{1}{\lambda} e^{(\mu - \delta) \bar{T}} \left[1 - e^{-\lambda z} \right] - \frac{1}{\kappa} \right) + \frac{x}{\kappa} - C_b z - e^{-\delta \bar{T}} C_s z, \end{aligned}$$

which is strictly positive for all $z > 0$ provided \bar{T} is large enough. \square

5. The infinite time horizon case ($\bar{T} = \infty$). Throughout this section, we write $v(x, y)$ instead of $v(\infty, x, y)$ and we assume that

$$(5.1) \quad \mu < \delta \quad \text{and} \quad C_s > 0.$$

If $\mu \leq \delta$ and $C_s = 0$, then selling all available shares at time 0 is optimal (see Lemma 2.(II)). If $\mu = \delta$ and $C_s > 0$, then the value function is the same as the value function corresponding to the case $\mu = \delta$ and $C_s = 0$ and selling all available shares at time T gives rise to a sequence of ε -optimal strategies because

$$\frac{1}{\lambda} \mathbb{E} \left[e^{-\delta T} X_T^0 \right] \left[1 - e^{-\lambda y} \right] - e^{-\delta T} C_s y = \frac{1}{\lambda} x \left[1 - e^{-\lambda y} \right] - e^{-\delta T} C_s y \xrightarrow{T \rightarrow \infty} \frac{1}{\lambda} x \left[1 - e^{-\lambda y} \right].$$

Furthermore, if $\mu > \delta$, then the value function is identical to ∞ and selling all available shares at time T gives rise to a sequence of ε -optimal strategies because

$$\frac{1}{\lambda} \mathbb{E} \left[e^{-\delta T} X_T^0 \right] \left[1 - e^{-\lambda y} \right] - e^{-\delta T} C_s y = \frac{1}{\lambda} x e^{(\mu - \delta)T} \left[1 - e^{-\lambda y} \right] - e^{-\delta T} C_s y \xrightarrow{T \rightarrow \infty} \infty.$$

In light of the heuristics we considered in the previous section that explain the structure of the HJB equation (4.1)–(4.2) and the fact that buying shares is not part of the optimal tactics when $\mu < \delta$ (see Lemma 2.(I)), we solve the stochastic control problem that arises when $\bar{T} = \infty$ and (5.1) holds true by constructing an appropriate solution $w : \mathbb{R}_+^* \times \mathbb{R}_+ \rightarrow \mathbb{R}$ to the HJB equation

$$(5.2) \quad \max \{ \mathcal{L}w(x, y), -\lambda x w_x(x, y) - w_y(x, y) + x - C_s \} = 0,$$

where \mathcal{L} is defined by (4.3), with boundary condition

$$(5.3) \quad w(x, 0) = 0 \quad \text{for all } x > 0.$$

To this end, we look for a solution w to (5.2)–(5.3) that is characterised by a function $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that partitions the state space $\mathbb{R}_+^* \times \mathbb{R}_+$ into two regions, the “waiting” region \mathcal{W} and the “selling” region \mathcal{S} , defined by

$$(5.4) \quad \mathcal{W} = \{ (x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+ \mid y > 0 \text{ and } x < F(y) \} \cup (\mathbb{R}_+^* \times \{0\}),$$

$$(5.5) \quad \mathcal{S} = \{ (x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+ \mid y > 0 \text{ and } x \geq F(y) \},$$

Inside \mathcal{W} , w should satisfy the differential equation

$$\frac{1}{2} \sigma^2 x^2 w_{xx}(x, y) + \mu x w_x(x, y) - \delta w(x, y) = 0.$$

The only solution to this ODE that remains bounded as $x \downarrow 0$ is given by

$$(5.6) \quad w(x, y) = A(y) x^n,$$

for some function $A : \mathbb{R}_+ \rightarrow \mathbb{R}$, where n is the positive solution to the quadratic equation

$$(5.7) \quad \frac{1}{2} \sigma^2 \ell(\ell - 1) + \mu \ell - \delta \equiv \frac{1}{2} \sigma^2 \ell^2 + \left(\mu - \frac{1}{2} \sigma^2 \right) \ell - \delta = 0.$$

For future reference, we note that $n > 1$ if and only if $\delta > \mu$. On the other hand, w should satisfy

$$(5.8) \quad -\lambda x w_x(x, y) - w_y(x, y) + x - C_s = 0, \quad \text{for } (x, y) \in \mathcal{S},$$

which implies that

$$(5.9) \quad -\lambda x w_{xx}(x, y) - \lambda w_x(x, y) - w_{yx}(x, y) + 1 = 0, \quad \text{for } (x, y) \in \mathcal{S}.$$

To proceed further, we look for A and F such that w is $C^{2,1}$. To this end, it turns out that it suffices to require that the derivatives w_y and w_{yx} are continuous along the free-boundary F . Such a requirement, (5.6) and (5.8)–(5.9) yield the system of equations

$$\begin{aligned} -\lambda n A(y) x^n - \dot{A}(y) x^n + x - C_s \Big|_{x=F(y)} &= 0, \\ -\lambda n^2 A(y) x^{n-1} - n \dot{A}(y) x^{n-1} + 1 \Big|_{x=F(y)} &= 0, \end{aligned}$$

which is equivalent to

$$(5.10) \quad F(y) = \frac{n C_s}{n-1} =: F_\circ,$$

$$(5.11) \quad \dot{A}(y) F_\circ^n = -\lambda n A(y) F_\circ^n + F_\circ - C_s.$$

In view of the boundary condition (5.3) and (5.6), we require that $A(0) = 0$ and we solve (5.11) to obtain

$$(5.12) \quad A(y) = e^{-\lambda n y} \int_0^y e^{\lambda n u} \frac{1}{n} \left(\frac{n-1}{n C_s} \right)^{n-1} du = \frac{1}{\lambda n^2} \left(\frac{n-1}{n C_s} \right)^{n-1} (1 - e^{-\lambda n y}).$$

The analysis thus far has fully characterised w inside the waiting region \mathcal{W} . To determine w inside the selling region \mathcal{S} , we consider the function \mathbb{Y} defined by

$$(5.13) \quad \mathbb{Y}(x) = \frac{1}{\lambda} \ln \frac{x}{F_\circ}, \quad \text{for } x > 0,$$

and we note that

$$(5.14) \quad F_\circ - x = -x \left[1 - e^{-\lambda \mathbb{Y}(x)} \right] \quad \text{and} \quad y - \mathbb{Y}(x) > 0 \Leftrightarrow x < F_\circ e^{\lambda y}.$$

In particular, we note that the restriction of \mathbb{Y} in (F_\circ, ∞) partitions the selling region into

$$\begin{aligned} \mathcal{S}_1 &= \{(x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \mid x \geq F_\circ \text{ and } y \leq \mathbb{Y}(x)\}, \\ \mathcal{S}_2 &= \{(x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \mid x \geq F_\circ \text{ and } y > \mathbb{Y}(x)\} \end{aligned}$$

(see also Figure 1). The region \mathcal{S}_1 is the part of the state space where it is optimal to sell all available shares at time 0. On the other hand the region \mathcal{S}_2 is the part of the state space where it is optimal to sell an amount $\mathbb{Y}(x)$ of shares at time 0 and then sell continuously in a manner such that the optimal joint process (X^*, Y^*) is reflected in the line $x = F_\circ$ in an appropriate oblong way until all shares are exhausted. These considerations and the structure of the performance criterion that we maximise suggest that

$$\begin{aligned} w(x, y) &= \frac{1}{\lambda} x \left[1 - e^{-\lambda y} \right] - C_s y, \quad \text{if } (x, y) \in \mathcal{S}_1, \\ w(x, y) &= w(F_\circ, y - \mathbb{Y}(x)) + \frac{1}{\lambda} x \left[1 - e^{-\lambda \mathbb{Y}(x)} \right] - C_s \mathbb{Y}(x), \quad \text{if } (x, y) \in \mathcal{S}_2. \end{aligned}$$

We conclude this discussion with the candidate for a solution to the HJB equations (5.2)–(5.3) given by

$$(5.15) \quad w(x, y) = \begin{cases} 0, & \text{if } y = 0 \text{ and } x > 0, \\ A(y)x^n, & \text{if } y > 0 \text{ and } x \leq F_\circ, \\ A(y - \mathbb{Y}(x))F_\circ^n + \frac{x - F_\circ}{\lambda} - C_s \mathbb{Y}(x), & \text{if } y > 0 \text{ and } F_\circ < x < F_\circ e^{\lambda y}, \\ \frac{1}{\lambda} x [1 - e^{-\lambda y}] - C_s y, & \text{if } y > 0 \text{ and } F_\circ e^{\lambda y} \leq x. \end{cases}$$

We can now prove the main result of the section, which shows that this function is indeed the control problem's value function and identifies an optimal liquidation strategy.

Proposition 5. *Suppose that $\bar{T} = \infty$ and that (5.1) holds true. The function w defined by (5.15), where F_\circ , A are given by (5.10), (5.12), is a $C^{2,1}$ solution to the HJB equation (5.2) that identifies with the control problem's value function, namely,*

$$(5.16) \quad v(x, y) = \sup_{\xi^s \in \mathcal{A}_{\infty, y}^s} I_{\infty, x, y}(\xi^s, 0) = w(x, y) \quad \text{for all } (x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+.$$

If we define

$$(5.17) \quad \xi_t^{s*} = y \wedge \sup_{0 \leq s \leq t} \frac{1}{\lambda} [\ln x + B_s - \ln F_\circ]^+, \quad \text{for } t > 0,$$

where

$$(5.18) \quad B_t = \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t,$$

then the following statements are true:

- (I) If $\mu - \frac{1}{2} \sigma^2 \geq 0$, then $(\xi^{s*}, 0)$ is an optimal liquidation strategy.
- (II) If $\mu - \frac{1}{2} \sigma^2 < 0$, then $(\xi^{s*}, 0)$ is not an admissible liquidation strategy. In this case, if we define

$$(5.19) \quad \xi_t^{s*j} = \xi_t^{s*} \mathbf{1}_{\{t \leq j\}} + y \mathbf{1}_{\{j < t\}}, \quad \text{for } t > 0 \text{ and } j \geq 1,$$

then $(\xi^{s*j}, 0)$ gives rise to a sequence of ε -optimal strategies.

Proof. In view of its construction, we will prove that w is $C^{2,1}$ if we show that w_y , w_x and w_{xx} are continuous along the free-boundary F as well as along the restriction of \mathbb{Y} in (F_\circ, ∞) . To this end, we consider any $(x, y) \in \mathcal{S}_2$ and we use the ODE (5.11) that A satisfies as well as the definition (5.13) of \mathbb{Y} to calculate

$$(5.20) \quad w_y(x, y) = \dot{A}(y - \mathbb{Y}(x))F_\circ^n,$$

$$(5.21) \quad w_x(x, y) = \left[-\dot{A}(y - \mathbb{Y}(x))F_\circ^n - C_s \right] \frac{1}{\lambda x} + \frac{1}{\lambda}$$

$$= nA(y - \mathbb{Y}(x)) \frac{F_\circ^n}{x} + \frac{1}{\lambda} \left[1 - \frac{F_\circ}{x} \right]$$

and

$$\begin{aligned}
w_{xx}(x, y) &= -n\dot{A}(y - \mathbb{Y}(x))F_\circ^n \frac{1}{\lambda x^2} - nA(y - \mathbb{Y}(x)) \frac{F_\circ^n}{x^2} + \frac{F_\circ}{\lambda x^2} \\
&= n(n-1)A(y - \mathbb{Y}(x)) \frac{F_\circ^n}{x^2} - \frac{1}{\lambda x^2} [(n-1)F_\circ - nC_s] \\
(5.22) \quad &= n(n-1)A(y - \mathbb{Y}(x)) \frac{F_\circ^n}{x^2},
\end{aligned}$$

where the last identity follows thanks to (5.10). These calculations imply the required continuity results along F because $\lim_{x \downarrow F_\circ} \mathbb{Y}(x) = 0$. Also, these calculations, the observation that

$$\dot{A}(0) = \lim_{y \downarrow 0} \dot{A}(y) = (F_\circ - C_s)F_\circ^{-n},$$

which follows from (5.11) and the fact that $A(0) = 0$, imply that given any point $x > F_\circ$ and any sequence $(x_n, y_n) \in \mathcal{S}_2$ converging to $(x, \mathbb{Y}(x))$,

$$\begin{aligned}
\lim_{n \rightarrow \infty} w_y(x_n, y_n) &= F_\circ - C_s, \\
\lim_{n \rightarrow \infty} w_x(x_n, y_n) &= \frac{1}{\lambda} \left[1 - \frac{F_\circ}{x} \right] = \frac{1}{\lambda} \left[1 - e^{-\lambda \mathbb{Y}(x)} \right] \quad \text{and} \quad \lim_{n \rightarrow \infty} w_{xx}(x_n, y_n) = 0
\end{aligned}$$

These expressions are the same as the corresponding ones that we derive using the definition (5.15) of w for a point $x > F_\circ$ and any sequence $(x_n, y_n) \in \mathcal{S}_1$ converging to $(x, \mathbb{Y}(x))$, and the required continuity results along the restriction of \mathbb{Y} in (F_\circ, ∞) follow.

By the construction and the $C^{2,1}$ continuity of w , we will show that w satisfies the HJB equation (5.2) if we prove that

$$(5.23) \quad -\lambda x w_x(x, y) - w_y(x, y) + x - C_s \leq 0 \quad \text{for all } (x, y) \in \mathcal{W}.$$

$$(5.24) \quad \frac{1}{2} \sigma^2 x^2 w_{xx}(x, y) + \mu x w_x(x, y) - \delta w(x, y) \leq 0 \quad \text{for all } (x, y) \in \mathcal{S}.$$

In view of (5.11), we can see that (5.23) is equivalent to

$$\frac{x - C_s}{x^n} \leq \frac{F_\circ - C_s}{F_\circ^n} \quad \text{for all } x \leq F_\circ,$$

which is true thanks to the calculation

$$\frac{d}{dx} \left(\frac{x - C_s}{x^n} \right) = \frac{n-1}{x^{n+1}} \left(\frac{nC_s}{n-1} - x \right) > 0 \quad \text{for all } x < F_\circ = \frac{nC_s}{n-1}.$$

To prove (5.24), we first note that the quadratic equation (5.7), which $n > 1$ satisfies, and the definition of F_\circ in (5.10) imply that

$$(5.25) \quad \delta C_s - (\delta - \mu)F_\circ = -\frac{C_s(\delta - \mu n)}{n-1} = -\frac{1}{2} \sigma^2 n C_s < 0.$$

Given any $(x, y) \in \mathcal{S}_1$, we use the fact that $x > F_\circ e^{\lambda y}$ to calculate

$$(5.26) \quad \begin{aligned} \frac{1}{2}\sigma^2 x^2 w_{xx}(x, y) + \mu x w_x(x, y) - \delta w(x, y) &= -\frac{\delta - \mu}{\lambda} x \left[1 - e^{-\lambda y}\right] + \delta C_s y \\ &\leq -\frac{\delta - \mu}{\lambda} F_\circ \left[e^{\lambda y} - 1\right] + \delta C_s y =: Q_1(y). \end{aligned}$$

Also, we use the fact that n satisfies (5.7) to see that, given any $(x, y) \in \mathcal{S}_2$,

$$(5.27) \quad \frac{1}{2}\sigma^2 x^2 w_{xx}(x, y) + \mu x w_x(x, y) - \delta w(x, y) = -\frac{\delta - \mu}{\lambda} (x - F_\circ) + \delta C_s \mathbb{Y}(x) =: Q_2(x).$$

In view of (5.25) and the definition (5.13) of \mathbb{Y} , we can see that

$$Q_1(0) = 0 \quad \text{and} \quad Q'_1(y) = \delta C_s - (\delta - \mu) F_\circ e^{\lambda y} < 0 \quad \text{for all } y \geq 0,$$

and

$$Q_2(F_\circ) = 0 \quad \text{and} \quad Q'_2(x) = -\frac{\delta - \mu}{\lambda} + \frac{\delta C_s}{\lambda x} \leq \frac{\delta C_s - (\delta - \mu) F_\circ}{\lambda F_\circ} < 0 \quad \text{for all } x \geq F_\circ.$$

It follows that the right-hand side of (5.26) (resp., (5.27)) is negative for all $y \geq 0$ (resp., $x \geq F_\circ$), and (5.24) has been established.

We have established the first identity in (5.16) in Lemma 2.(I). To derive the second one, we consider any strategy $\xi^s \in \mathcal{A}_{\infty, y}^s$. Arguing in exactly the same way as in the proof of Theorem 3 up to (4.18)–(4.19) and using the positivity of w instead of the inequality (4.10), we can show that

$$(5.28) \quad \int_{[0, T]} e^{-\delta t} (X_t - C_s) \circ_\lambda d\xi_t^s \leq w(x, y) + M_T.$$

where

$$(5.29) \quad M_T = \sigma \int_0^T e^{-\delta t} X_t w_x(X_t, Y_t) dW_t.$$

This result and (3.3) in Lemma 1 imply that the random variable $\inf_{t \in [0, T]} M_t$ is bounded from below by an integrable random variable for any given $T > 0$. Therefore, the stochastic integral M is a supermartingale. In light of this observation, we can take expectations in (5.28) to obtain

$$I_{\infty, x, y}(\xi^s, 0) = \limsup_{T \rightarrow \infty} J_{T, x, y}(\xi^s, 0) = \limsup_{T \rightarrow \infty} \mathbb{E} \left[\int_{[0, T]} e^{-\delta t} (X_t - C_s) \circ_\lambda d\xi_t^s \right] \leq w(x, y).$$

It follows that $v(x, y) \leq w(x, y)$ because $\xi^s \in \mathcal{A}_{\infty, y}^s$ has been arbitrary.

To prove the reverse inequality and establish the optimality of the processes ξ^{s^*} given by (5.17) and $\xi^{b^*} = 0$, we note that apart from a jump of size $\frac{1}{\lambda} \left(\ln \frac{x}{F_\circ} \right)^+ = (\mathbb{Y}(x))^+$ at time 0,

the process $(\ln x + B - \lambda \xi^{s*}, Y^* - \xi^{s*})$ is reflecting in the line $x = \ln F_\circ$ in the direction defined by the vector $(-\lambda, -1)$. In particular,

$$\ln x + B_t - \lambda \xi_t^{s*} \leq \ln F_\circ \quad \text{and} \quad \xi_t^{s*} - \xi_0^{s*} = \int_{]0,t[} \mathbf{1}_{\{\ln x + B_t - \lambda \xi_t^{s*} = \ln F_\circ\}} d\xi_s^{s*} \quad \text{for all } t \leq \tau^*,$$

where $\tau^* = \inf\{t \geq 0 \mid \xi_t^{s*} = y\}$. In view of this observation and (2.6), if we denote by X^* and Y^* the price process and the remaining amount of shares process associated with $(\xi^{s*}, 0)$, then

$$(X_t^*, Y_t^*) \in \mathcal{W} \quad \text{and} \quad \xi_{t+}^{s*} = \int_{[0,t]} \mathbf{1}_{\{(X_s^*, Y_s^*) \in \mathcal{S}\}} d\xi_s^{s*} \quad \text{for all } t \geq 0,$$

where the waiting region \mathcal{W} and the selling region \mathcal{S} are given by (5.4) and (5.5). Also, we can check that the strategy $(\xi^{s*}, 0)$ is admissible provided that $\lim_{T \rightarrow \infty} Y_T^* = 0$. In view of (5.17)–(5.18), we can see that this is indeed the case if and only if $\mu - \frac{1}{2}\sigma^2 \geq 0$ because a Brownian motion with negative drift has supremum over time that is an exponentially distributed random variable.

In the same way as in the proof of Theorem 3, we now see that

$$(5.30) \quad \int_{[0,T]} e^{-\delta t} [(X_t^* - C_s) \circ_\lambda d\xi_t^{s*}] + e^{-\delta T} w(X_{T+}^*, Y_{T+}^*) = w(x, y) + M_T^*$$

where the local martingale M^* is defined as in (5.29). In view of this identity, (3.3) in Lemma 1 and the inequality

$$(5.31) \quad 0 \leq w(X_t^*, Y_t^*) \leq \frac{1}{\lambda n^2} \left(\frac{n-1}{nC_s} \right)^{n-1} F_\circ^n,$$

which follows from (5.12) and the definition (5.15) of w , we can see that the random variable $\sup_{t \in [0, T]} M_t$ is bounded from above by an integrable random variable for all $T > 0$. Therefore, M^* is a submartingale, and we can take expectations in (5.30) to obtain

$$(5.32) \quad \begin{aligned} J_{T,x,y}(\xi^{s*}, 0) &= \mathbb{E} \left[\int_{[0,T]} e^{-\delta t} [(X_t^* - C_s) \circ_\lambda d\xi_t^{s*}] \right] \\ &= w(x, y) - \mathbb{E} \left[e^{-\delta T} w(X_{T+}^*, Y_{T+}^*) \right]. \end{aligned}$$

These identities and (5.31) imply that

$$I_{\infty,x,y}(\xi^{s*}, 0) = \limsup_{T \rightarrow \infty} J_{T,x,y}(\xi^{s*}, 0) \geq w(T, x, y).$$

Combining this result with the inequality $v(x, y) \leq w(x, y)$ that we have established above, we derive (5.16) as well as the optimality of $(\xi^{s*}, 0)$, which is admissible if and only if $\mu - \frac{1}{2}\sigma^2 \geq 0$.

If $\mu - \frac{1}{2}\sigma^2 < 0$, then we can use (5.32) to check that the strategy $(\xi^{s^*j}, 0)$ given by (5.19) has payoff

$$\begin{aligned} I_{\infty, x, y}(\xi^{s^*j}, 0) &= \limsup_{T \rightarrow \infty} J_{T, x, y}(\xi^{s^*j}, 0) \\ &= \mathbb{E} \left[\int_{[0, j]} e^{-\delta t} [(X_t^* - C_s) \circ_{\lambda} d\xi_t^{s^*}] \right] + \frac{1}{\lambda} x e^{-(\delta - \mu)j} \left[1 - e^{-\lambda(y - Y_j^*)} \right] \\ &= w(x, y) - \mathbb{E} \left[e^{-\delta j} w(X_{j+}^*, Y_{j+}^*) \right] + \frac{1}{\lambda} x e^{-(\delta - \mu)j} \left[1 - e^{-\lambda(y - Y_j^*)} \right]. \end{aligned}$$

The inequality $v(x, y) \leq w(x, y)$ and the fact that the right-hand side of this expression converges to $w(x, y)$ as $j \rightarrow \infty$ imply (5.16) and establish that $(\xi^{s^*j}, 0)$ is a sequence of ε -optimal strategies.

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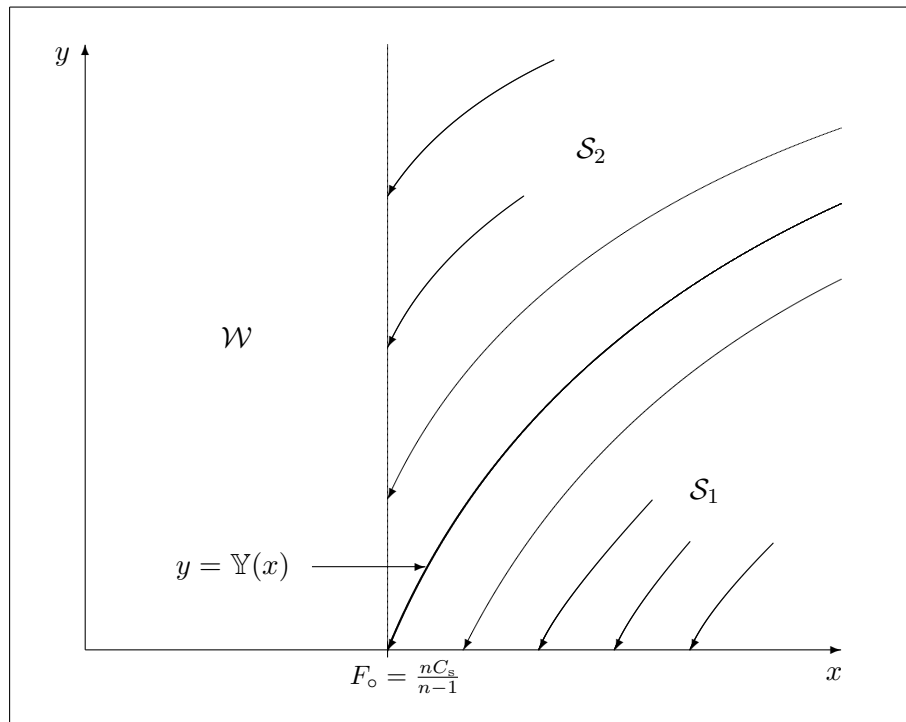


Figure 1. The regions providing the optimal strategy when $\bar{T} = \infty$. If the stock price takes values in the “waiting” region \mathcal{W} , then it is optimal to take no action. If the stock price at time 0 is inside the “selling” region \mathcal{S}_1 , then it is optimal to sell all available shares immediately. If the stock price at time 0 is inside the “selling” region \mathcal{S}_2 , then it is optimal to liquidate an amount that would cause the stock price to drop to F_0 and then keep on selling until all shares are exhausted by just preventing the stock price to rise above F_0 .